TROY UNIVERSITY | CENTER FOR RELATIVITY AND COSMOLOGY

Kerr Geometry: Lecture Notes

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Contents

1 Introductory Remarks

These notes are intended to accompany the video lectures on Kerr Geometry, which are available here. The content closely follows the classic text on Kerr geometry by Barrett O'Neill [1], which serves as the primary reference throughout. Please note that O'Neill's notations for the spacetimes K^* and $^*K$ are represented in these notes as $\bar{K}$ and K^*, respectively. Additionally, the discussion on energy extraction is drawn directly from Chandrasekhar's authoritative text, The Mathematical Theory of Black Holes [2]. The only prerequisite for these notes is a basic understanding of general relativity at the level of an introductory textbook. Finally, if you notice any errors or lack of clarity, please reach out to me at and I will modify these notes accordingly.

2 Begining Kerr Geometry

The Kerr metric describes the time-independent, axis-symmetric gravitational field of a collapsed object that has retained its angular momentum. All matter having collapsed, the Kerr metric satisfies the vacuum Einstein equation given by $R_{\mu\nu} = 0$. In Boyer-Lindquist coodinate system (t, r, θ, φ), the Kerr metric takes the form

$$ds^2 = g_{tt}\, c^2 dt^2 + 2\, g_{t\varphi}\, dt\, d\varphi + g_{rr}\, dr^2 + g_{\theta\theta}\, d\theta^2 + g_{\varphi\varphi}\, d\varphi^2 \;, \tag{1}$$

where

$$g_{tt} = -1 + \frac{2Mr}{\rho^2}, \quad g_{t\varphi} = \frac{-2Mra\sin^2\theta}{\rho^2}, \quad g_{rr} = \frac{\rho^2}{\Delta}\;, \quad g_{\theta\theta} = \rho^2, \quad g_{\varphi\varphi} = \frac{\Sigma^2 \sin^2\theta}{\rho^2}\;. \tag{2}$$

Here

$$\rho^2 = r^2 + a^2\cos^2\theta\;, \quad \Delta = r^2 - 2Mr + a^2\;, \;\text{ and }\; \Sigma^2 = (r^2+a^2)^2 - \Delta\, a^2 \sin^2\theta\;.$$

As we shall see, M can be interpreted as the mass, and aM the angular momentum of the black hole. The metric coefficient functions are independent of t and φ as expected from the assumed symmetry. When $a \to 0$, the Kerr metric reduces to the Schwarzschild metric. The Schwarzschild metric is both static and spherically symmetric, and consequently describes the end product of a non-rotating spherically symmetric collapse. In the expressions above, and as in the remainder of these notes, I have set $c = G = 1$.

The contravariant form of the Kerr metric tensor is given by

$$g = -\frac{\Sigma^2}{\rho^2\Delta}\partial_t\otimes\partial_t - \frac{2Mar}{\rho^2\Delta}\partial_t\otimes\partial_\varphi - \frac{2Mar}{\rho^2\Delta}\partial_\varphi\otimes\partial_t + \frac{\Delta}{\rho^2}\partial_r\otimes\partial_r + \frac{1}{\rho^2}\partial_\theta\otimes\partial_\theta + \frac{(\Delta - a^2\sin^2\theta)}{\rho^2\Delta\sin^2\theta}\partial_\varphi\otimes\partial_\varphi\;. \tag{3}$$

We will find the following relationships obeyed by the components of the Kerr metric in Boyer-Lindquist coordinates useful. Since they can be easily verified by algebraic manipulation, we state them without proof. For the covariant form of the metric

$$\begin{aligned}
a\sin^2\theta g_{tt} + g_{t\varphi} &= -a\sin^2\theta,\\
(r^2+a^2)g_{t\varphi} + a g_{\varphi\varphi} &= a\sin^2\theta\Delta,\\
(r^2+a^2)g_{tt} + a g_{t\varphi} &= -\Delta,\\
a\sin^2\theta g_{t\varphi} + g_{\varphi\varphi} &= (r^2+a^2)\sin^2\theta.
\end{aligned} \tag{4}$$

And for the contravariant form

$$\begin{aligned}
\Delta\, g^{tt} - a\; \sin^2\theta\, \Delta\, g^{t\varphi} &= -(r^2+a^2)\\
a g^{tt} - (r^2+a^2)\, g^{t\varphi} &= -a\\
\Delta g^{t\varphi} - a\; \sin^2\theta \Delta\, g^{\varphi\varphi} &= -a\\
-a\sin^2\theta g^{t\varphi} + (r^2+a^2)\sin^2\theta g^{\varphi\varphi} &= 1\;.
\end{aligned} \tag{5}$$

Note

$$g^{\varphi\varphi} = \frac{1}{\Delta\sin^2\theta}\left[1 - \frac{2Mr}{\rho^2}\right]\;, \;\text{ and }\; \sqrt{-g} = \rho^2\sin\theta\;.$$

2.1 Singularities of K

1. In the BL coordinates since $-\det g = \rho^2 \sin\theta$, the poles defined by $\theta = 0$ and $\theta = \pi$ are singular points. As can be expected, these singularities will disappear in Cartesian-like coordinates.

2. There is a physical singularity when $\rho^2 = 0$ which deserves some attention. This happens only when $r = 0$ and $\theta = \pi/2$. If this is the case, it is not clear whether $r = 0$ is truly a point. To understand the nature of the region $r = 0$, consider spacelike curves of type

$$\alpha(\varphi) = (t_0, r = 0, \theta_0, \varphi)$$

when $\theta \neq \pi/2$. These curves have proper length

$$L = \int_0^{2\pi} \sqrt{g_{\varphi\varphi}}\,|_{r=0}\; d\varphi = \int_0^{2\pi} a \sin\theta_0 \; d\varphi = 2\pi\; a \sin\theta_0\; .$$

I.e., when $\theta \neq \pi/2$, the region $r = 0$ has structure since the curve α has length $2\pi a$ in the limit $\theta = \pi/2$. Consequently, $\rho^2 = 0$ is the famous ring singularity (RS) of the Kerr geometry defined by

$$\mathbb{RS} = \{r = 0,\ \theta = \pi/2\}\; .$$

On $\mathbb{RS}$, the scalar curvature $R_{abcd}R^{abcd} \to \infty$ as can be easily verified.

3. K has a coordinate singularity when $\Delta = 0$. This happens when

$$r = r_\pm \equiv M \pm \sqrt{M^2 - a^2}\; .$$

The surfaces $r = r_\pm$ are called Horizons (H_+ and H_- respectively). As suggested by their description as a coordinate singularity, and as we shall see, they can be transformed away using cleverly chosen coordinates.

Much like the spherical coordinate system in Minkowski and Schwarzschild spacetime, the polar axis, given by $\theta = 0$ and $\theta = \pi$, is a coordinate singularity. This is easily seen by noting that at the polar axis $\sqrt{-g} = 0$, making the metric degenerate in the Boyer-Lindquist coordinate system. In section 2.8, we will explicitly coordinate transform the polar singularities away. With this in mind, we will treat the polar axis as a meaningful region of our spacetime. The Kerr metric given in eq.(1) is asymptotically flat provided we have the usual interpretation for the Boyer-Lindquist coordinates as r approaches infinity. I.e., we set $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\,\pi$. In particular we require that φ is a cyclic coordinate (φ_0 and $\varphi_0 + 2\pi$ locates the same point).

Definition 1. *The Kerr metric in the Boyer-Lindquist coordinate system extended to the axis $\{\theta = 0,\ \pi\}$ is referred to as the spacetime K.*

Definition 2. *The two horizons H_- and H_+ divide K into three open regions. Block I is the region defined by $r > r_+$, Block II is the region defined by $r_- < r < r_+$, and Block III is the region defined by $r < r_-$.*

2.2 In Falling Kerr-Schild Coordinates

Problem 1. *Define vector fields l and n by*

$$l = \frac{1}{\Delta}\Big[(r^2 + a^2)\partial_t + \Delta\partial_r + a\partial_\varphi\Big]$$

and

$$n = \frac{1}{\Delta}\Big[(r^2 + a^2)\partial_t - \Delta\partial_r + a\partial_\varphi\Big]$$

in the Boyer-Lindquist coordinate system. Show that n and l are null geodesic vector fields in the Boyer-Lindquist coordinate system.

Definition 3. *n and l respectively will be referred to as the infalling and outgoing null geodesics of the Kerr geometry.*

Just like in the case of the Schwarzschild metric, we will show that that a simple coordinate transformation will remove the singularities at $H_{\pm}$. Naturally, for this to happen, the coordinate transformations will have to be singular at $r = r_{\pm}$. The infalling Kerr-Schild coordinates are $(\bar{t}, \bar{r}, \bar{\theta}, \bar{\varphi})$. They are related to the Boyer-Lindquist coordinates by the following relations:

$$\bar{r} = r\ , \quad \bar{\theta} = \theta\ , \quad d\bar{t} = dt + \frac{r^2 + a^2}{\Delta} dr\ , \quad \text{and} \quad d\bar{\varphi} = d\varphi + \frac{a}{\Delta} dr. \tag{6}$$

The "barred" is placed on r and θ so that no confusions arise while performing coordinate transformations. We will have plenty of opportunities to compare various components of tensors in the Boyer-Lindquist and Kerr-Schild coordinates. Therefore, it will be crucial to establish the transformation properties as early as possible. Clearly,

$$\begin{bmatrix} d\bar{t} \\ d\bar{r} \\ d\bar{\theta} \\ d\bar{\varphi} \end{bmatrix} = \begin{bmatrix} 1 & G & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & H & 0 & 1 \end{bmatrix} \begin{bmatrix} dt \\ dr \\ d\theta \\ d\varphi \end{bmatrix}, \tag{7}$$

where

$$G = \frac{r^2 + a^2}{\Delta} \qquad \text{and} \qquad H = \frac{a}{\Delta}.$$

We can write the above equation as

$$d\bar{x}^{\mu} = A^{\mu}{}_{\nu}\, dx^{\nu}\ ,$$

where $A^{\mu}{}_{\nu}$ is the transformation matrix defined in eq.(7), and

$$d\bar{x}^{\mu} = (d\bar{t}, d\bar{r}, d\bar{\theta}, d\bar{\varphi})\ .$$

The lower indices refer to columns, and the upper indices refer to rows.

Problem 2. *Show that the components of the dual vectors transforms as*

$$\begin{pmatrix} w_{\bar{t}} \\ w_{\bar{r}} \\ w_{\bar{\theta}} \\ w_{\bar{\varphi}} \end{pmatrix} = \left(A^{-1}\right)^T \begin{pmatrix} w_t \\ w_r \\ w_\theta \\ w_\varphi \end{pmatrix} \tag{8}$$

where

$$A^{-1} = \begin{bmatrix} 1 & -G & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -H & 0 & 1 \end{bmatrix}.$$

Problem 3. *Show that the basis vector transform as*

$$\begin{pmatrix} \partial_{\bar{t}} \\ \partial_{\bar{r}} \\ \partial_{\bar{\theta}} \\ \partial_{\bar{\varphi}} \end{pmatrix} = (A^{-1})^T \begin{pmatrix} \partial_t \\ \partial_r \\ \partial_\theta \\ \partial_\varphi \end{pmatrix}.$$

Problem 4. *Show that the components of a vector Y transforms as*

$$\begin{pmatrix} Y^{\bar{t}} \\ Y^{\bar{r}} \\ Y^{\bar{\theta}} \\ Y^{\bar{\varphi}} \end{pmatrix} = (A) \begin{pmatrix} Y^t \\ Y^r \\ Y^\theta \\ Y^\varphi \end{pmatrix}. \tag{9}$$

Eqs.(8) and (9) can be used to transform general tensors. We are now in a position to compute the metric tensor in the Kerr-Schild coordinate system. Various metric identities listed in eq.(4) will be required to simplify the expressions. To illustrate the nature of the simplifications, will carry out the calculation of $g_{\bar{r}\bar{r}}$ explicitly, leaving the others to the reader to verify.

$$g_{\bar{r}\bar{r}} = (A^{-1})^{\alpha}{}_{1}\,(A^{-1})^{\beta}{}_{1}\,g_{\alpha\beta} = [G^2 g_{tt} + GH g_{t\varphi}] + [GH g_{t\varphi} + H^2 \gamma_{\varphi\varphi}] + g_{rr}.$$

But,

$$G^2 g_{tt} + GH g_{t\varphi} = \frac{G}{\Delta}\Big[2Mr g_{tt} + a g_{t\varphi}\Big] = \frac{G}{\Delta}\Big[(r^2+a^2) g_{tt} + a g_{t\varphi}\Big].$$

Using eq.(4), the above equation becomes

$$\Big[G^2 g_{tt} + GH g_{t\varphi}\Big] = -G\ .$$

Similary,

$$\Big[GH g_{t\varphi} + H^2 \gamma_{\varphi\varphi}\Big] = \frac{a^2\sin^2\theta}{\Delta}.$$

From the above calculations we get that

$$g_{\bar{r}\bar{r}} = 0\ .$$

In a similar manner, we find that in Kerr-Schild coordinates, the metric components in the basis $(\bar{t}, \bar{r}, \bar{\theta}, \bar{\varphi})$ become

$$g_{\mu\nu} = \begin{bmatrix} z-1 & 1 & 0 & -za\sin^2\theta \\ 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & \rho^2 & 0 \\ -za\sin^2\theta & -a\sin^2\theta & 0 & \Sigma^2\sin^2\theta/\rho^2 \end{bmatrix}, \tag{10}$$

where $z = 2Mr/\rho^2$. Since $\det A = 1$, here $\sqrt{-g} = \rho^2 \sin\theta$, as well.

From mere inspection, we see that

Lemma 1. *The Kerr-Schild metric in eq. (10) does not suffer from a coordinate singularity at* $\Delta = 0$.

Definition 4. *The Kerr metric in the Kerr-Schild coordinate system, extended to the axis* $\{\theta = 0,\ \pi\}$ *is referred to as the infalling Kerr-Schild spacetime and is denoted by* $\bar{K}$.

Definition 5. *The two horizons* H_- *and* H_+ *divide* $\bar{K}$ *into three open regions. Block* $\bar{I}$ *is the region defined by* $\bar{r} > r_+$*, Block* $\bar{I}\bar{I}$ *is the region defined by* $r_- < \bar{r} < r_+$*, and Block* $\bar{I}\bar{I}\bar{I}$ *is the region defined by* $\bar{r} < r_-$.

2.3 Time Orientation of the Infalling Kerr-Schild Spacetime

As we shall see, the causal character of the coordinate function t changes even outside the outer horizon H_+. Therefore, to understand the causal structure of $\bar{K}$, it is crucial to consistently choose future pointing light cones if possible. From the form of the Kerr-Schild metric in eq.(10), it is clear that

$$-\partial_{\bar{r}}$$

is a smooth, nowhere vanishing, lightlike vector-field that is well defined on $\bar{K}$.

Definition 6. *We shall pick* $-\partial_{\bar{r}}$ *as future pointing in* $\bar{K}$*, thus making* $\bar{K}$ *a time oriented spacetime.* [1]

Problem 5. *Show that, outside the horizons* $H_\pm$*,*

$$-\partial_{\bar{r}} = n = \frac{1}{\Delta}\Big[(r^2+a^2)\partial_t - \Delta\partial_r + a\partial_\varphi\Big]\ .$$

Naturally, we require that as $r \to \infty$, ∂_t is future pointing, so that far away from the singularity RS, life goes on as usual, and time increases in the future direction. So, the time orientation of $\bar{K}$ must contain ∂_t for large r.

[1] Recall what this means. A causal vector X, which is not proportional to $-\partial_{\bar{r}}$ in $\bar{K}$ is future pointing if and only if $g(X, -\partial_{\bar{r}}) < 0$. If X is proportional to $-\partial_{\bar{r}}$ then the proportionality constant must be greater the zero for X to be future pointing (and evidently null). This is easily verified in a local orthonormal frame where the null vector has the appearance $(1, 1, 00)$.

Problem 6. *In the Boyer-Lindquist coordinate system, show that as r approaches infinity, ∂_t is future pointing timelike.*

Hence our choice of future is consistent with our expectations of how time flows as $r \to \infty$. Furthermore, in $\bar{K}$, by n we will mean the vector field $-\partial_{\bar{r}}$. In the Boyer-Lindquist coordinate system the expression for n is given by definition 1.

2.4 Graviational Redshift

Definition 7. $g_{tt} = 0$ *when*

$$r = r_{\infty\,\pm}(\theta) = M \pm \sqrt{M^2 - a^2\cos^2\theta}\,. \tag{11}$$

$r = r_{\infty\,-}$ *and* $r = r_{\infty\,+}$ *are the called infinite redshift surfaces.*

The reason for the definition above will now be made clear. Consider a static observer at some fixed $r = r_1 > r_{\infty\,+}$. Such an observer has four velocity

$$u_1 = \frac{1}{\sqrt{-g_{tt}(r_1)}}\partial_t\,.$$

Here,

$$-g_{tt}(r_1) \equiv 1 - \frac{2Mr_1}{r_1^2 + a^2\cos^2\theta}$$

is the relevant metric coefficient evaluated at r_1. Suppose this observer were to send a photon of energy E (as measured at $r = \infty$), along the outgoing radial null geodesic l. Such a photon would have a 4-momentum given by

$$p^\mu = \frac{E}{\Delta}\left[(r^2 + a^2)\partial_t + \Delta\partial_r + a\partial_\varphi\right]\,.$$

The frequency of the emitted photon ν_1 as measured by the observer at r_1 is given by the expression

$$h\,\nu_1 = -g(p, u_1)$$

$$= -\frac{E}{\Delta}\left[(r_1^2 + a^2)g_{tt}(r_1)\frac{1}{\sqrt{-g_{tt}(r_1)}} + a\frac{1}{\sqrt{-g_{tt}(r_1)}}g_{t\varphi}(r_1)\right]$$

$$= -\frac{E}{\Delta\sqrt{-g_{tt}(r_1)}}\left[(r_1^2 + a^2)g_{tt}(r_1) + ag_{t\varphi}(r_1)\right]\,.$$

Therefore, from eq.(4) we get that

$$h\,\nu_1 = \frac{E}{\sqrt{-g_{tt}(r_1)}}\,.$$

This photon if received by another observer at $r = r_2$, has a frequency

$$h\nu_2 = \frac{E}{\sqrt{-g_{tt}(r_2)}}\,.$$

Then, since the energy of the photon E is constant along the geodesic [2].

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{tt}(r_1)}{g_{tt}(r_2)}}\,.$$

Thus, the frequency of the received photon decreases as it moves away from the black hole (see figure 1). This is an example of gravitational redshift. Clearly, as $r_1 \to r_{\infty\,+}\,,\quad \nu_2 \to 0$. This is why we call $r_{\infty\,+}$ an infinite redshift surface. Similar remarks apply to the surface $r_{\infty\,-}$ if we pick $0 < r_2 < r_1 < r_{\infty\,-}$, and let $r_1 \to r_{\infty\,-}$. Clearly, the infinite redshift surfaces intersect the horizons at the poles.

[2]If you are not familiar with this argument, we will show this explicitly when we study the geodesic equation in Kerr geometry.

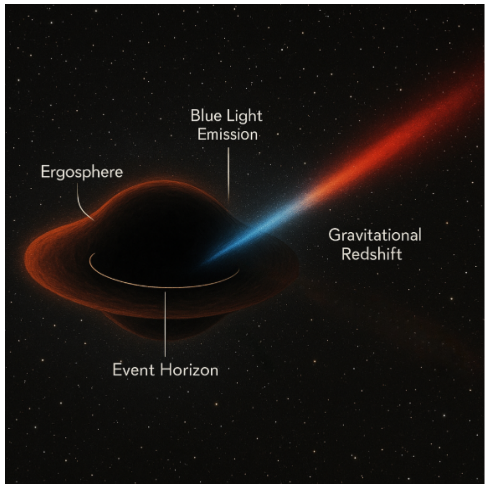

Figure 1: Gravitational reshift from a beam of light emitted from near $r_{\infty\,+}$. In the next section, we will see that $r_{\infty\,+}$ forms a boundary of the outer ergosphere.

2.5 The Ergospheres

In the late sixties, Vishveshwara [3] pointed out that there are regions in $\bar{K}$ where ∂_t ceases to be timelike (even outside the horizon). This feature brings about new and interesting physics. It is easily verified that $g_{tt} > 0$ in the region bounded by $r_{\infty\,-}$ and $r_{\infty\,+}$. Therefore, in

$$D = \{r : r_{\infty\,-} < r < r_{\infty\,+}\}\ ,$$

∂_t is spacelike.

Definition 8. *$\epsilon_+ = \{\, D\ \cap$ Block $\bar{\mathrm{I}}\,\}$ is the outer Ergosphere, and, $\epsilon_- = \{\, D\ \cap$ Block $\overline{\mathrm{III}}\,\}$ is the inner Ergosphere. $\epsilon = \epsilon_-\ \cup\ \epsilon_+$ is the Ergosphere.*

Amusingly, $\boxed{\partial_t,\ \partial_r,\ \partial_\theta,\ \partial_\varphi \text{ are spacelike in the ergosphere!}}$

Lemma 2. *In the ergosphere ϵ, $\nabla^\mu t$ [3] is past-directed and timelike.*

Proof. To see this, note that eq.(3) implies that

$$g_{\mu\nu}\nabla^\mu t \nabla^\nu t = g^{\mu\nu}\nabla_\mu t \nabla_\nu t = g^{tt} < 0\ .$$

Therefore, $\nabla^\mu t$ is timelike in the ergosphere. Since n is future pointing in the ergosphere,

$$g_{\mu\nu} l^\mu \nabla^\nu t = l^\mu \nabla_\mu t = l^0 = \frac{r^2 + a^2}{\Delta} > 0$$

implies that $\nabla^\mu t$ is past-directed timelike in the ergosphere. □

Let $\alpha(\lambda)$ be the path taken by a causal particle in ergosphere, such that $u = \dot{\alpha}$ is its 4-velocity. As we shall see, even though ∂_t is not timelike in the ergosphere, the coordinate function t does increase for causal particles in the ergosphere.

Lemma 3. *In the ergosphere, for a causal particle, $\dot{t} > 0$.*

[3] Recall that

$$\nabla^\mu t = g^{\mu\nu}\nabla_\nu t = g^{\mu\nu}\partial_\nu t\ .$$

Proof. Since $\nabla^\mu t$ is past directed and timelike, and u is future pointing and causal

$$0 < g_{\mu\nu} u^\mu \nabla^\nu t = u^\mu \nabla_\mu t = \dot{t} \,.$$

□

In the ergosphere, since ∂_t is spacelike, an observer cannot remain static. A *static* observer is one whose curve traced out in the spacetime has a fixed value of r, θ, and φ. At best, all we can have are stationary observers. Stationary observers move along constant values of r and θ. The required rotation of an observer in the ergosphere can be thought of as an extreme case of *frame dragging.*

Theorem 1. *As a causal particle falls into the ergosphere, it starts to rotate along with the spacetime.*

Proof. Let $\alpha(\tau)$ be the curve traced out the by a stationary observer. The four-velocity of the observer then takes the form

$$\dot{\alpha} = u(\tau) = (c\dot{t}, 0, 0, \dot{\varphi}).$$

We also require that

$$-1 = u^2 = \left[g_{tt}\, \dot{t}^2 + g_{\varphi\varphi}\, \dot{\varphi}^2 + 2\, g_{t\varphi} \dot{t}\, \dot{\varphi} \right] .$$

If such an observer is to be static, $\dot{\varphi}$ must vanish. Inside the ergosphere, all but the last term on the right hand side of the above equation is positive. Therefore, for the above equation to hold true

$$g_{t\varphi}\, \dot{\varphi} < 0$$

for timelike curve since $\dot{t} > 0$ even in the ergosphere. From eq.(2) the above inequality remains true only when

$$a\, \dot{\varphi} > 0.$$

It is easily see that this even true for photons in the ergosphere. Since, we need that

$$0 = u^2 = \left[g_{tt} c^2\, \dot{t}^2 + g_{\varphi\varphi}\, \dot{\varphi}^2 + 2\, g_{\varphi}\, c\dot{t}\, \dot{\varphi} \right] .$$

□

In particular, there are no static observers in the ergosphere, for the observer is forced to rotate along with the spacetime.

2.6 The Black Hole Region of Infalling Kerr-Schild Geometry

It is important to remember, that while we may freely interchange between the Boyer-Lindquist coordinates (ct, r, θ, φ) and the Kerr-Schild coordinates $(\bar{t}, \bar{r}, \bar{\theta}, \bar{\varphi})$, the time orientation is fixed by the vector $-\partial_{\bar{r}}$.

Problem 7. *Show that in the region $r_- < r < r_+$, $-\partial_r$ is future pointing and timelike.*

For a particle with future pointing 4-velocity u,

$$g(u, -\partial_r) < 0 \quad \rightarrow \quad \frac{\rho^2}{|\Delta|}\, \dot{r} < 0 \quad \rightarrow \quad \dot{r} < 0$$

when $r_- < r < r_+$. Therefore, if a particle (massless or otherwise) enters the region $r < r_+$, it will have necessarily have to move along decreasing values of r until it is thrown into the region $r < r_-$ where Δ is positive and $-\partial_r$ is no longer timelike. Therefore $r = r_+$ forms a one-way membrane, and is the *event horizon* of the Kerr-Schild geometry. Similarly, particles in $r < r_-$ may never enter the region $r > r_-$. The region $r_- < r < r_+$ is referred to as the *Black Hole region of the Kerr-Schild geometry.* $r = r_-$ is also referred to as a horizon.

Problem 8. *Show that in Block $\overline{II}$, i.e., the black hole region, $-l$ is future pointing, and so it falls inward.*

Definition 9. *The infalling Kerr-Schild spacetime $\bar{K}$ will also be referred to as the Kerr black hole spacetime.*

From the problem above, we see that even the outgoing beam of light is drawn in by the black hole.

2.7 The Geometry of the Kerr-Schild Horizons

The horizons are where the Boyer-Lindquist coordinate system fails. So we will carry out our analysis of the horizon in the Kerr-Schild coordinates. The horizons are located at $\bar{r} = r_\pm$ and is denoted by $H_\pm$. Since the coordinate $\bar{r}$ is fixed, we say that $H_\pm$ is a 3-dimensional submanifold of the geometry. The tangent space of $H_\pm$ is spanned by $(\partial_{\bar{t}}, \partial_{\bar{\theta}}, \partial_{\bar{\varphi}})$.

Problem 9. *Show that,*

$$V_\pm = (r_\pm^2 + a^2)\partial_{\bar{t}} + a\partial_{\bar{\varphi}}$$

is a future-pointing null vector in $H_\pm$ outside of the poles. At the poles however

$$V_\pm = (r_\pm^2 + a^2)\partial_{\bar{t}}$$

is a null vector.

Problem 10. *Show that $V_\pm$ as defined above is orthogonal to every tangent vector in $H_\pm$ outside of the poles.*

Isn't that curious! While $V_\pm$ belongs to the tangent space of $H_\pm$, it is also orthogonal to every tangent vector in $H_\pm$ which includes itself. For this reason we say the $H_\pm$ is a null hypersurface.

Theorem 2. *The integral curve of $V_\pm$ is a null pre-geodesic in $\bar{K}$ that lies on the horizon.*

Proof. Consider the outgoing principal null geodesic tangent vector fields in Blocks I and III of K given by

$$l = \frac{1}{\Delta}\left[(r^2 + a^2)\partial_t + \Delta\partial_r + a\partial_\varphi\right] .$$

In these blocks, define a vector field

$$\tilde{l} = \frac{\Delta}{2}\, l\, .$$

Then in blocks I and III

$$\nabla_{\tilde{l}}\, \tilde{l} \;=\; \frac{\Delta}{2}\, \nabla_l \left(\frac{\Delta}{2} l\right) = \frac{1}{4}\,\Delta^2 \nabla_l\, l \;+\; \frac{1}{4}\,\Delta\,(\nabla_l\,\Delta)\, l = \;\frac{1}{4}\,\Delta\,(\nabla_l\,\Delta)\, l\, .$$

I.e.,

$$\nabla_{\tilde{l}}\tilde{l} = (r - M)\,\tilde{l}\, ,$$

and $\tilde{l}$ is a null pregeodesic in Blocks I and III. In Kerr-Schild coordinates,

$$\begin{aligned}\tilde{l} &= \left[(r^2 + a^2)\partial_{\bar{t}} + a\partial_{\bar{\varphi}}\right] + \Delta\left[G\frac{\partial}{\partial\bar{t}} + \frac{\partial}{\partial\bar{r}} + H\frac{\partial}{\partial\bar{\varphi}}\right] \\ &= (r^2 + a^2)\partial_{\bar{t}} + \Delta/2\;\partial_{\bar{r}} + a\partial_{\bar{\varphi}}\, .\end{aligned}$$

Since $\tilde{l}$ is a smooth, well defined vector field on the horizons we get that

$$\nabla_{\tilde{l}}\tilde{l}|_{H_\pm} = (r_\pm - M)\,\tilde{l}\, . \tag{12}$$

Therefore, on the horizon, $\nabla_{\tilde{l}}\tilde{l}$ is a null pre-geodesic, and, $\tilde{l}|_{H_\pm} = V_\pm$. □

Theorem 3. *$V_\pm$ is future pointing.*

Proof. On $H_\pm$,

$$g(V_\pm, n) = (z - 1)(r^2 + a^2) - z(r^2 + a^2) - za^2\sin^2\theta + a^2\sin^2\theta(1 + z) = -\rho^2 < 0\, .$$

□

Theorem 4. *Let u be a future-pointing causal vector at $r_{\pm}$. Then in Kerr-Schild coordinates,*

$$u^{\bar{r}} \leq 0 \ .$$

Equality is realized in the above equation iff $u = c\, V_{\pm}$ for some $c > 0$. Therefore, the entire future lightcone lies in the blackhole region side of H_+, and outside the blackhole region side of H_-.

Proof. Since $V_{\pm}$ is future pointing at $H_{\pm}$, when $u \neq V_{\pm}$, we have that $g(V_{\pm}, u) < 0$. I.e.,

$$0 > g(V_{\pm}, u) = u^{\bar{t}} \left[(r_{\pm}^2 + a^2)\; g_{\bar{t}\bar{t}} + a\; g_{\bar{t}\bar{\varphi}} \right]$$

$$+ u^{\bar{\varphi}} \left[(r_{\pm}^2 + a^2)\; g_{\bar{t}\bar{\varphi}} + a\; g_{\bar{\varphi}\bar{\varphi}} \right] + u^{\bar{r}} \left[(r_{\pm}^2 + a^2)\; g_{\bar{t}\bar{r}} + a\; g_{\bar{\varphi}\bar{r}} \right]$$

$$= \left[u^{\bar{t}}(-\Delta) + u^{\bar{\varphi}}(a \sin^2\theta \Delta) + u^{\bar{r}} \rho^2 \right] \Big|_{H_{\pm}} = u^{\bar{r}} \rho^2 \Big|_{H_{\pm}} \ .$$

Therefore, if $u \neq cV_{\pm}$, $u^{\bar{r}} < 0$. If $u = cV_{\pm}$, clearly, $u^{\bar{r}} = 0$. ☐

By now, it should be clear that in $\bar{K}$, at H_+, the future cone points "inward". Indeed, it is possible for a particle in Block I of $\bar{K}$ to escape to $r = \infty$.

Theorem 5. *p is a point in Block I of $\bar{K}$ if and only if there is future pointing timelike curve $\alpha(\tau)$ from p such that $r(\tau) \to \infty$ in the distant future.*

Proof. As we have already seen, if p is in either Block II or III, it is impossible for such a timelike curve to exist. Therefore, let p belong to Block I. We will deform the outgoing null geodesic vector l to get a necessary timelike curve.

Let $\alpha(\tau)$ be a curve such that $\alpha(0) = p$, and

$$\frac{dt}{d\tau} = \frac{r^2 + a^2}{\Delta} \ , \quad \frac{dr}{d\tau} = \lambda \ , \quad \frac{d\theta}{d\tau} = 0 \ , \quad \text{and} \quad \frac{d\varphi}{d\tau} = \frac{a}{\Delta} \ .$$

Here, λ is a constant such that $0 < \lambda < 1$. It is easy to verify that

$$g(\dot{\alpha},\ \dot{\alpha}) = (\lambda^2 - 1)\ \frac{\rho^2}{\Delta} < 0 \ .$$

It is also easy to verify that $\dot{\alpha}$ is future pointing. By setting limits on λ, we have effectively slowed down a null geodesic with $\dot{r} = 1$. ☐

Figure 2 is a depiction of the light cones at the horizons.

2.8 Removing The Polar Singularities

We will now construct a coordinate system for $\bar{K}$ which will be well defined on the poles ($\theta = 0, \pi$). Anticipating this coordinate system we had included the poles in $\bar{K}$ even while using the Kerr-Schild coordinate system. In the case of Minkowski spacetime, the polar singularity of the the spherical coordinate system is removed by going into the Cartesian coordinate system (although usually, the Cartesian coordinate system is the natural starting point). Here, too, we will want to go to a "Cartesian" like coordinate system. The new coordinates are labeled (T, X, Y, Z). Here we will have that

$$X^2 + Y^2 = (\bar{r}^2 + a^2) \sin^2 \bar{\theta} \ . \tag{13}$$

For then $\bar{r} = 0$, $\theta =$ constant are circles of radius $a \sin\theta$, revealing the proper length of the singularity (in the $\theta \to \pi/2$ limit naturally). This is accomplished by the transformations

$$X = (\bar{r} \cos\bar{\varphi} - a \sin\bar{\varphi}) \sin\bar{\theta} \ ,$$

and

$$Y = (\bar{r} \sin\bar{\varphi} + a \cos\bar{\varphi}) \sin\bar{\theta} \ .$$

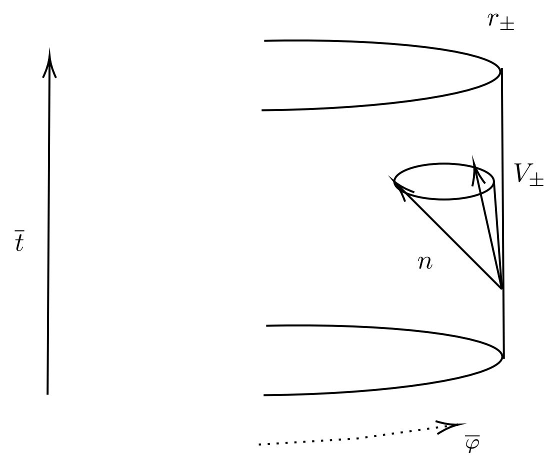

Figure 2: Geometry of the Kerr horizon. The light cone points inward at $r_\pm$. The future light cone intersects the tangent space of the horizon at exactly $V_\pm$. Notice that $V_\pm$ has a $\bar{t}$ and $\bar{\varphi}$ component.

Just like in the spherical coordinate system, we set

$$Z = \bar{r}\cos\theta\ , \quad \text{and} \quad T = \bar{t} - \bar{r}\ .$$

Then

$$\begin{pmatrix} \partial_{\bar{t}} \\ \partial_{\bar{r}} \\ \partial_{\bar{\theta}} \\ \partial_{\bar{\varphi}} \end{pmatrix} = A \begin{pmatrix} \partial_T \\ \partial_X \\ \partial_Y \\ \partial_Z \end{pmatrix}, \tag{14}$$

where the matrix A is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & \cos\bar{\varphi}\ \sin\bar{\theta} & \sin\bar{\varphi}\ \sin\bar{\theta} & \cos\bar{\theta} \\ 0 & X\ \cot\bar{\theta} & Y\ \cot\bar{\theta} & -\bar{r}\sin\bar{\theta} \\ 0 & -Y & X & 0 \end{pmatrix}. \tag{15}$$

Note that $\det A = \rho^2 \sin\bar{\theta}$ is singular at the poles. This is in fact, necessary to remove the polar singularities. By inverting the matrix above, we get

$$\begin{pmatrix} \partial_T \\ \partial_X \\ \partial_Y \\ \partial_Z \end{pmatrix} = A^{-1} \begin{pmatrix} \partial_{\bar{t}} \\ \partial_{\bar{r}} \\ \partial_{\bar{\theta}} \\ \partial_{\bar{\varphi}} \end{pmatrix},$$

where

$$A^{-1} = \frac{1}{\rho^2} \begin{pmatrix} \rho^2 & 0 & 0 & 0 \\ \bar{r}X & \bar{r}X & X\ \cot\bar{\theta} & A_{13} \\ \bar{r}Y & \bar{r}Y & Y\ \cot\bar{\theta} & A_{23} \\ (\bar{r}^2 + a^2)\cos\bar{\theta} & (\bar{r}^2 + a^2)\cos\bar{\theta} & -\bar{r}\sin\bar{\theta} & a\cos\bar{\theta} \end{pmatrix}. \tag{16}$$

Here

$$A_{13} = -(\bar{r}\sin\bar{\varphi} + a\cos\bar{\varphi}\ \cos^2\bar{\theta})/\sin\bar{\theta}$$

and

$$A_{23} = (\bar{r}\cos\bar{\varphi} - a\sin\bar{\varphi}\ \cos^2\bar{\theta})/\sin\bar{\theta}\ .$$

The dual bases transform as

$$\begin{pmatrix} d\bar{t} \\ d\bar{r} \\ d\bar{\theta} \\ d\bar{\varphi} \end{pmatrix} = (A^{-1})^T \begin{pmatrix} dT \\ dX \\ dY \\ dZ \end{pmatrix}.$$

$(A^{-1})^T$ is easily obtained from eq.(16). The components of vectors now transform as

$$\begin{pmatrix} V^{\bar{t}} \\ V^{\bar{r}} \\ V^{\bar{\theta}} \\ V^{\bar{\varphi}} \end{pmatrix} = (A^{-1})^T \begin{pmatrix} V^T \\ V^X \\ V^Y \\ V^Z \end{pmatrix},$$

and similarly, the components of dual vectors transform as

$$\begin{pmatrix} w_{\bar{t}} \\ w_{\bar{r}} \\ w_{\bar{\theta}} \\ w_{\bar{\varphi}} \end{pmatrix} = A \begin{pmatrix} w_T \\ w_X \\ w_Y \\ w_Z \end{pmatrix}.$$

It is possible to obtain the $\bar{K}$ metric in the Kerr-Schild-Polar coordinates by the usual method, but there is a more enlightening/efficient method. Consider the dual of the infalling null geodesic vector field

$$-n^\flat = (\partial_{\bar{r}})^\flat = (\ g_{\bar{t}\bar{r}},\ g_{\bar{r}\bar{r}},\ g_{\bar{\theta}\bar{r}},\ g_{\bar{\varphi}\bar{r}}\ ) = (\ 1,\ 0,\ 0,\ -a\sin^2\bar{\theta}\ ).$$

Or

$$-n^\flat = d\bar{t} - a\sin^2\bar{\theta}\ d\bar{\varphi}.$$

It is easy to check that the metric in eq.(10) can be written as

$$g = \Big[- d\bar{t}\otimes d\bar{t} + d\bar{t}\otimes d\bar{r} + d\bar{r}\otimes d\bar{t} + (r^2+a^2)\sin^2\bar{\theta} d\bar{\varphi}\otimes d\bar{\varphi} + \rho^2 d\bar{\theta}\otimes d\bar{\theta} - a\sin^2\bar{\theta} d\bar{r}\otimes d\bar{\varphi} - a\sin^2\bar{\theta} d\bar{\varphi}\otimes d\bar{r}\Big] + zn^\flat\otimes n^\flat\ .$$

After some algebra, it is easily shown that the terms in the square Brackets is the Minkowski metric:

$$\Big[- d\bar{t}\otimes d\bar{t} + d\bar{t}\otimes d\bar{r} + d\bar{r}\otimes d\bar{t} + (r^2+a^2)\sin^2\bar{\theta} d\bar{\varphi}\otimes d\bar{\varphi} + \rho^2 d\bar{\theta}\otimes d\bar{\theta} - a\sin^2\bar{\theta} d\bar{r}\otimes d\bar{\varphi} - a\sin^2\bar{\theta} d\bar{\varphi}\otimes d\bar{r}\Big]$$

$$= \Big[- dT\otimes dT + d\bar{r}\otimes d\bar{r} + (r^2+a^2)\sin^2\bar{\theta} d\bar{\varphi}\otimes d\bar{\varphi} + \rho^2 d\bar{\theta}\otimes d\bar{\theta} - a\sin^2\bar{\theta} d\bar{r}\otimes d\bar{\varphi} - a\sin^2\bar{\theta} d\bar{\varphi}\otimes d\bar{r}\Big]$$

$$= \Big[- dT\otimes dT\ +\ dX\otimes dX\ +\ dY\otimes dY\ +\ dZ\otimes dZ\Big].$$

On the other hand

$$n^\flat = d\bar{t} - a\sin^2\bar{\theta} d\bar{\varphi} = \frac{1}{\rho^2}\Big[\rho^2 dT + rXdX + rYdY + (r^2+a^2)\cos\bar{\theta} dZ\Big]$$

$$+\frac{a\sin\bar{\theta}}{\rho^2}\Big[(r\sin\bar{\varphi} + a\cos\bar{\varphi}\cos^2\bar{\theta}) - (r\cos\bar{\varphi} - a\sin\bar{\varphi}\cos^2\bar{\theta})dY - a\sin\bar{\theta}\cos\bar{\theta} dZ\Big].$$

Collecting and simplifying, we get

$$n^\flat = \Big[dT + \frac{r(XdX+YdY) - a(XdY-YdX)}{r^2+a^2} + \frac{Z}{r}dZ\Big].$$

Consequently,

$$g = -dT^2 + dX^2 + dY^2 + dZ^2 + \frac{2Mr}{\rho^2}\Big[dT + \frac{r(XdX+YdY) - a(XdY-YdX)}{r^2+a^2} + \frac{Z}{r}dZ\Big]^2,$$

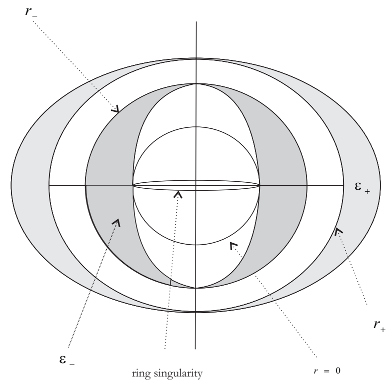

Figure 3: The shaded regions are the ergospheres. The central ring indicates the true singularity of the Kerr geometry. The innermost circle is the surface $r = 0$, which is described by the eq.(13). The interior of this region will be taken up in the following section. Also see Figure 6.

Or finally in the (T, X, Y, Z) coordinate system the metric of $\bar{K}$ is given by

$$g = -dT^2 + dX^2 + dY^2 + dZ^2 + \frac{2M\bar{r}^3}{\bar{r}^4 + a^2\ Z^2}\left[dT + \frac{r(XdX + YdY) - a(XdY - YdX)}{r^2 + a^2} + \frac{Z}{r}dZ\right]^2 . \qquad (17)$$

Here, $\bar{r}$ is determined (modulo sign) by the implicit equation

$$\bar{r}^4 - (X^2 + Y^2 + Z^2 - a^2)\,\bar{r}^2 - a^2\ Z^2 = 0\ .$$

Since $\bar{K}$ spacetime is what we are concerned about, we first fix $\bar{r}$ and then solve for T, X, Y, Z. Also, from eq.(17), the metric is well defined on the axis of the Kerr geometry.

Figure 3 shows some of the interesting regions we have discussed in the notes.

2.9 The Domain of the Radial Coordinate and the Dual Axis

The argument presented here is taken directly from [5]. Consider the expression

$$g_{00} = -1 + \frac{2Mr}{r^2 + a^2\cos^2\theta}$$

in the Boyer-Lindquist coordinates. If we are to interpret g_{00} as functions in the usual sense along the z axis, we get that

$$g_{00}(z) = -1 + \frac{2M|z|}{z^2}\ .$$

A plot of this expression is shown in Figure 4. This function is not differentiable when $r = 0$ and $\theta = 0, \pi$. Hence, this interpretation leads to a singularity in the curvature tensor when $r = 0$ and $\theta = 0, \pi$. That is curious indeed, considering the fact that the metric tensor is well defined at these locations. If we proceed in this manner, the Kerr metric is sourced by a thin disk of concentrated matter. These ideas have been explored in the literature. For example see [4]. However, if we do not want any avoidable singularities, there is another way forward. Suppose we allow r to take values from $-\infty$ to ∞, the expression for g_{00} becomes

$$g_{00}(r) = -1 + \frac{2Mr}{r^2}$$

for all real values of r. A plot of this expression is shown in Figure 5, thus rendering it smooth across $r = 0$. Hence, we take the topology of $\bar{K}$ to be

$$\mathbb{R}^2(\bar{t}, \bar{r}) \times \mathbb{S}^2(\bar{\theta}, \bar{\varphi}) - \mathbb{RS}\ ,$$

where the coordinate ranges are given by

$$-\infty < \bar{t} < \infty\ , \quad -\infty < \bar{r} < \infty\ , \quad 0 \leq \bar{\theta} \leq \pi\ , \quad 0 \leq \bar{\varphi} < 2\pi\ .$$

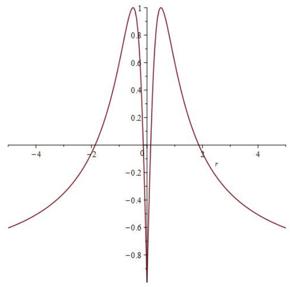

Figure 4: g_{00} along the z axis in the usual interpretation of a spherical coordinate system.

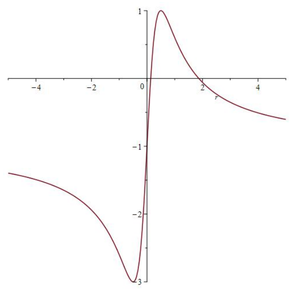

Figure 5: g_{00} along $r \in (-\infty, \infty)$.

Then we are back to the case when the Riemann tensor is only singular when $\rho^2 = 0$. [4]

There is a price to pay for such extensions. For example, the $\theta = 0$ symmetry axis runs from r equals $-\infty$ to ∞. A similar second copy of the symmetry axis exists for $\theta = \pi$ (see figure 6).

Definition 10. *Henceforth block $\bar{I}\bar{I}\bar{I}$ of $\bar{K}$ is the region $-\infty < \bar{r} < r_-$.*

3 Maximal Slow-Kerr Geometry

Slow means $M^2 > a^2$. We will restrict ourselves to this case. [1] has a description of the extreme case as well. This happens when $M^2 = a^2$. When $a^2 > M^2$, the original Boyer-Lindquist spacetime is already maximal. In this case, there are no horizons, and the geometry possesses a naked singularity.

In the last chapter, we expanded the Kerr geometry to include negative values of r. You may wonder why we should consider a further extension. To this end, consider the infalling null geodesic $n = -\partial_{\bar{r}}$ in $\bar{K}$. The geodesic curve of this vector field starts at $r \to \infty$ and crosses the horizons and either ends up at the ring singularity, or enters the $r < 0$ universe and will eventually reach $r \to -\infty$. However, the outgoing geodesic l in block $\bar{I}$ while it proceeds to $r \to \infty$, in the past cannot be extended to $r - r_+$ (since nothing can emerge outward from the event horizon). Mind you, this is not a problem for physics since one could argue that some cause had to source the photon just outside the event horizon. Nevertheless, mathematically, such an extension is possible where the outgoing geodesic will have emerged from a “white hole”. We will explore such possibilities in this chapter. In such a maximal spacetime, all causal geodesics can either be extended to all values of the affine parameter or end up in a singularity.

[4] Consider a sample component of the Riemann tensor given below.

$$R^{\bar{t}}_{\theta\theta,\bar{\varphi}} = -3Mar\sin^2(\theta)(r^2 + a^2)\frac{(r^2 - 3a^2\cos^2(\theta))}{(\rho^2)^3} \ .$$

This expression is only singular when $r = 0$ and $\theta = \pi/2$. The singularities in the other components of the Riemann tensor have similar characteristics.

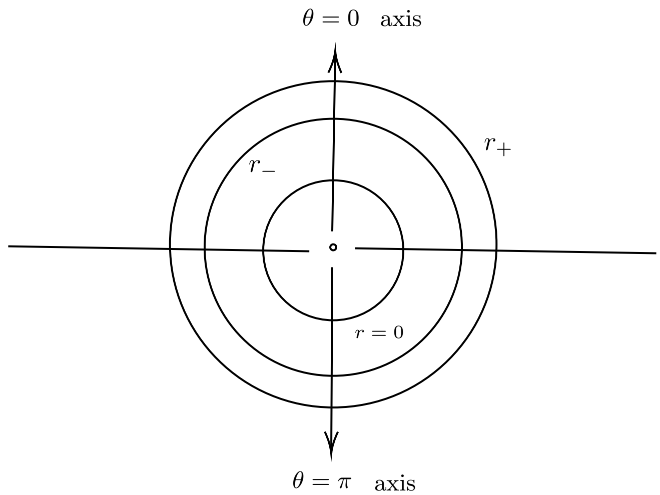

Figure 6: The radial coordinate is scaled at e^r. Inside $r = 0$ we have the region $r < 0$. The origin of the diagram corresponds to $e^{-\infty}$ and hence does not belong to $\bar{K}$. Hence, the disconnected symmetry axis.

3.1 The Nature of Kerr-Schild Coordinates

Problem 11. *Show that, up to an integration constant,*

$$T(r) \equiv \int \frac{r^2 + a^2}{\Delta}\, dr = r \;+\; \frac{r_+^2 + a^2}{r_+ - r_-} \ln |r - r_+| \;-\; \frac{r_-^2 + a^2}{r_+ - r_-} \ln |r - r_-| \;. \tag{18}$$

Problem 12. *Show that, up to an integration constant,*

$$A(r) \equiv \int \frac{a}{\Delta}\, dr = \frac{a}{r_+ - r_-} \;\ln \left| \frac{r - r_+}{r - r_-} \right| \;. \tag{19}$$

From the above two problems, we find that

$$\begin{aligned} \bar{t} &= t + T(r), \\ \lim_{r \to \pm\infty} T(r) &= \mp\infty, \\ \bar{\varphi} &= \varphi + A(r), \\ \lim_{r \to r_\pm} A(r) &= \pm\infty \;. \end{aligned} \tag{20}$$

3.2 The Outgoing Kerr-Schild Spacetime

The Outgoing Kerr-Schild Spacetime $K^\star$ are given by coordinates $(t^\star, r^\star, \theta^\star, \varphi^\star)$. They are related to the Boyer-Lindquist coordinates by the transformation

$$r^\star = r, \quad \theta^\star = \theta, \quad dt^\star = dt - \frac{r^2 + a^2}{\Delta} dr, \quad \text{and} \quad d\varphi^\star = d\varphi - \frac{a}{\Delta} dr. \tag{21}$$

"$\star$" is placed on r and θ so that no confusions arise while performing coordinate transformations. The transformations of vectors and forms are generated as before in the $\bar{K}$ spacetime, except that in the transformation matrix A, the functions G and H are replaced by

$$G = -\frac{r^2 + a^2}{\Delta} \qquad \text{and} \qquad H = -\frac{a}{\Delta}. \tag{22}$$

In $K^\star$, the spacetime metric in coordinates $(t^\star, r^\star, \theta^\star, \varphi^\star)$ become

$$g^\star_{\mu\nu} = \begin{bmatrix} z-1 & -1 & 0 & -za\sin^2\theta \\ -1 & 0 & 0 & a\sin^2\theta \\ 0 & 0 & \rho^2 & 0 \\ -za\sin^2\theta & a\sin^2\theta & 0 & \Sigma^2\sin^2\theta/\rho^2 \end{bmatrix}. \tag{23}$$

Since det $A = 1$, once again

$$\sqrt{-g^\star} = \rho^2 \sin\theta.$$

Note

$$t^\star = t - T(r),$$
$$\varphi^\star = \varphi - A(r). \tag{24}$$

It follows from mere inspection that the $K^\star$ metric in eq.(23) does not suffer from a coordinate singularity at $\Delta = 0$, and can be extended to the axes $\theta^\star = 0$ and $\theta^\star = \pi$ just as for the spacetime $\bar{K}$. Since the functions G and H only depend on r it is easily verified that on $K \cap\ K^\star$

$$\partial_{t^\star} = \partial_t\ ,$$
$$\partial_{\theta^\star} = \partial_\theta\ , \tag{25}$$
$$\partial_{\varphi^\star} = \partial_\varphi\ .$$

Theorem 6. *$K^\star$ is time orientable.*

Proof. Using the modified transformation matrix A, the null geodesic tangent vector field l in the $K^\star$ coordinate system becomes

$$l^{=}\partial_{r^\star}\ .$$

Therefore, $\partial_{r^\star}$ is a smooth, nowhere vanishing, lightlike vector field that is well defined on

$$\mathbb{R}^2(\bar{t}, \bar{r}) \times \mathbb{RS}^2(\bar{\theta}, \bar{\varphi}) - \mathbb{RS}\ .$$

□

Definition 11. *The lightcone containing $\partial_{r^\star}$ is defined to be future pointing in $K^\star$.*

As $r \to \infty$, since $g(\partial_{r^\star}, \partial_{t^\star}) = g(\partial_{r^\star}, \partial_t) = -1$ our choice of time orientation agrees with the flow of time for large values of $r^\star$. Here since causal curves flow out of the horizon, $K^\star$ describes a white hole.

Definition 12. *The two horizons H_- and H_+ divide $K^\star$ into three open regions. Block $I^\star$ is the region defined by $r^\star > r_+$, Block $II^\star$ is the region defined by $r_- < r^\star < r_+$, and Block $III^\star$ is the region defined by $r^\star < r_-$.*

Theorem 7. *In $K^\star$, $V^\star_\pm = (r^2_\pm + a^2)\partial_{t^\star} + a\partial_{\varphi^\star}$ is the future pointing null-pregeodesic generator of the horizons.*

Proof. The proof is similar to $\bar{K}$ case. It suffices to note that

$$g^\star(V^\star_\pm,\ \partial_{r^\star}) = -\rho^2 < 0\ .$$

□

Theorem 8. *In Block $II^\star$ of $K^\star$,*

$$-n = \partial_{r^\star} - \frac{2}{\Delta}[(r^2 + a^2)\partial_{t^\star} + a\partial_{\varphi^\star}]$$

is future pointing and null.

Proof. The proof is similar to $\bar{K}$ case. □

3.3 The Maximal Submanifold Around The Outer Horizon

In this section, we will want to cut and paste Boyer-Lindquist blocks with appropriate orientations of time so that geodesics that do not end in $\mathbb{RS}$ are complete. As a first step, we will require that all the principal null geodesics that do not end in $\mathbb{RS}$ are complete. There will be two distinct regions of interest. One that is centered around the region $r = r_+$, and the other that is centered around the other horizon $r = r_-$. Around each of these horizons, we will construct open submanifolds $D(r_\pm)$. The maximal geometry will consist of a sequence of alternating submanifolds $D(r_+)$ and $D(r_-)$.

Problem 13. *From problems 18 and 12, we see that*

$$A - \frac{a}{r_\pm^2 + a^2}\, T = -\frac{a}{r_\pm^2 + a^2}\Big[r + (r_\pm + r_\mp)\ln|r - r_\mp|\Big].$$

Consequently, $A - aT/(r_+^2 + a^2)$ is analytic when $r \neq r_-$, and $A - aT/(r_-^2 + a^2)$ is analytic when $r \neq r_+$.

Definition 13. *Define real valued functions U^+ and V^+ on $\bar{I}$ by*

$$\tan U^+ = \exp(-\kappa_+\, t^\star), \quad \tan V^+ = \exp(\kappa_+\, \bar{t})\ ,$$

and on $\bar{I}\bar{I}$ by

$$\tan U^+ = -\exp(-\kappa_+\, t^\star), \quad \tan V^+ = \exp(\kappa_+\, \bar{t})\ .$$

Also in $\bar{I}$, and $\bar{I}\bar{I}$, set

$$\theta^+ = \bar{\theta}\ , \quad \varphi^+ = \frac{1}{2}\Big[\bar{\varphi} + \varphi^\star - \frac{a}{r_+^2 + a^2}(\bar{t} + t^\star)\Big]\ .$$

Here,

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2(r_\pm^2 + a^2)}\ .$$

These functions are defined only on the Boyer-Lindquist blocks which do not include the horizons.

Note:

$$\frac{r_-^2 + a^2}{r_+^2 + a^2} = \frac{r_-}{r_+} = -\frac{\kappa_+}{\kappa_-}\ .$$

Theorem 9. *The functions $U^+, V^+, \theta^+, \varphi^+$ can be analytically extended to r_+.*

Proof. There is no need to elaborate on θ^+. Since $\bar{t}$ is well defined in $\bar{K}$, we have that $\tan V^+$ is well defined at r_+. In terms of $\bar{t}$

$$\tan U^+ = \operatorname{sgn}(r - r_+)\exp(-\kappa_+\bar{t})\exp(2\kappa_+ T(r))\ .$$

From the expression for $T(r)$ in problem 18 we find that

$$\tan U^+ = \operatorname{sgn}(r - r_+)\exp(-\kappa_+\bar{t})\exp(2\kappa_+ r)|r - r_+||r - r_-|^{-r_-/r_+}\ ,$$

but, $\operatorname{sgn}(r - r_+)|r - r_+| = (r - r_+)$, and $|r - r_-| = r - r_-$ in this patch, and so

$$\tan U^+ = (r - r_+)(r - r_-)^{-r_-/r_+}\ \exp[\kappa_+(2r - \bar{t})]\ . \tag{26}$$

Therefore, U^+ is well defined at r_+. Also, writing φ^+ in $\bar{K}$ coordinates we find that

$$\varphi^+ = \bar{\varphi} - \frac{a}{r_+^2 + a^2}\bar{t}\ - \Big[A(r) - \frac{a}{r_+^2 + a^2}T(r)\Big]\ .$$

From problem 13 we find that the term in the square brackets above is analytic for $r \neq r_-$. Therefore, φ^+ is well defined at r_+. □

Theorem 10. *The functions $U^+, V^+, \theta^+, \varphi^+$ form a coordinate system when $r > r_-$.*

Proof. Let

$$u^+ = \tan U^+ \quad \text{and} \quad v^+ = \tan V^+ \ . \tag{27}$$

Therefore, since we are not worried about orientation,

$$\begin{bmatrix} d\theta^+ \\ dv^+ \\ du^+ \\ d\varphi^+ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \kappa_+ v^+ & 0 & 0 \\ 0 & \partial_{\bar t} u^+ & \partial_{\bar r} u^+ & 0 \\ 0 & -\frac{a}{r_+^2+a^2} & -\left[\frac{dA}{dr} - \frac{a}{r_+^2+a^2}\,\frac{dT}{dr}\right] & 1 \end{bmatrix} \begin{bmatrix} d\bar\theta \\ d\bar t \\ d\bar r \\ d\bar\varphi \end{bmatrix} . \tag{28}$$

The determinant of the above map is

$$\kappa_+ \, (v^+) \, \frac{\partial(u^+)}{\partial \bar r} \ .$$

Since $v^+ \neq 0$ ($\bar t \neq -\infty$), $\{u^+, v^+, \theta^+, \varphi^+\}$ is a coordinate system so long as

$$\frac{\partial u^+}{\partial \bar r} \neq 0 \ .$$

From problem 14 below we have that

$$\frac{\partial u^+}{\partial \bar r} > 0 \ .$$

Since the tangent functions is a smooth invertible function on its principal domain, and since $(\tan U^+, \tan V^+, \theta^+, \varphi^+)$ is a valid coordinate system, so is $(U^+, V^+, \theta^+, \varphi^+)$. □

Problem 14. *From eq.(26) we see that*

$$u^+ = (r - r_+)(r - r_-)^{-r_-/r_+} \ \exp[\kappa_+(2r - \bar t)] \ .$$

Show that here

$$\frac{\partial u^+}{\partial \bar r} > 0 \ .$$

We will now consider the various coordinate limits of the boundary of $\bar I$ in the "bar" coordinate system. For example, from eq.24,

$$\tan U^+ = \exp(-\kappa_+ \, \bar t) \exp(2\kappa_+ T(r)) \ .$$

Therefore, as $r \to r_+$, eq.(20) implies that $U^+ \to 0$. In a similar manner we can compile the following list.

- As $\bar r \to r_+$, we have that $U^+ \to 0$ (included from above for completeness).
- As $\bar r \to \infty$, we have that $U^+ \to \pi/2$.
- As $\bar t \to \infty$, we have that $V^+ \to \pi/2$.
- As $\bar t \to -\infty$, we have that $V^+ \to 0$.

Note, as $\bar t \to \infty$, we could not have concluded that $U^+ \to 0$ since we have no information about $T(r)$.

Now we repeat the same analysis as above on $\bar I$ in the "star" coordinate system.

- As $r^\star \to r_+$, we have that $V^+ \to 0$.
- As $r^\star \to \infty$, we have that $V^+ \to \pi/2$.
- As $t^\star \to \infty$, we have that $U^+ \to 0$.
- As $t^\star \to -\infty$, we have that $U^+ \to \pi/2$.

Figure 7 contains much more information than we have covered so far, but for the moment, the limits on block $\bar{I}$ should agree with the list above. It is a simple exercise to verify that the limits on $\bar{I}\bar{I}$ is also as expected. Note that the regions defined by $V^+ = \pi/2$, $U^+ = \pm\pi/2$ does not belong to our spacetime, and for now, the region defined by $V^+ = 0$ has not yet been included in our analysis.

We will now create another copy of $\bar{I}$ and $\bar{I}\bar{I}$ which are exactly the same submanifolds as before. On these copies we will define U^+ and V^+ with the opposite sign from the previous definition 13. Later on, we will impose an opposite causal orientation on these sets. To distinguish the two set of submanifolds we denote these submanifolds as $\bar{I}'$ and $\bar{I}\bar{I}'$. To be clear $\bar{I}'$ is the same as $\bar{K}$ restricted to $\bar{r} > r_+$, and $\bar{I}\bar{I}'$ is the same as $\bar{K}$ restricted to $r_- < \bar{r} < r_+$.

Definition 14. *Define real valued functions U^+ and V^+ on $\bar{I}'$ by*

$$\tan U^+ = -\exp(-\kappa_+ \, t^\star), \quad \tan V^+ = -\exp(\kappa_+ \, \bar{t}) \ ,$$

and on $\bar{I}\bar{I}'$ by

$$\tan U^+ = \exp(-\kappa_+ \, t^\star), \quad \tan V^+ = -\exp(\kappa_+ \, \bar{t}) \ .$$

Also in $\bar{I}'$, and $\bar{I}\bar{I}'$, set

$$\theta^+ = \theta \ , \quad \varphi^+ = \frac{1}{2}\Big[\bar{\varphi} + \varphi^\star - \frac{a}{r_+^2 + a^2}(\bar{t} + t^\star)\Big] \ .$$

Here,

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2(r_\pm^2 + a^2)} \ .$$

These functions are defined only on the Boyer-Lindquist blocks, which do not include the horizons.

The analogs of theorems 9 and 10 continue to hold here. Also, exactly in the same way as in the previous case, the various labels on the boundaries of blocks $\bar{I}'$ and $\bar{I}\bar{I}'$ can be completed in a similar manner and is listed in figure 7.

Just as $t^\star \to \infty$, $\bar{r} = r_+$ when $V^+ > 0$ was included using the extended coordinate system $(U^+, V^+, \bar{\theta}, \varphi^+)$ in theorem 10 using $\bar{K}$ spacetime, we can include the horizons $t^\star \to \infty$, $\bar{r} = r_+$ when $V^+ < 0$ using $\bar{K}$ in figure 7. Similarly, using $K^\star$ coordinates, we can include the horizons when $V^+ = 0$ in figure 7.

To conclude, we have discussed all the features in figure 7 save two items: The labelled vector fields $n, l, -n, l$ and their importance, and the region described by $U^+ = 0 = V^+$. This will be taken up in the following sections.

3.4 The Orientation of Time Around The Outer Horizon

Definition 15. *We have already seen that*

- *in block $\bar{I}$, n, l is future-pointing null.*
- *in block $\bar{I}\bar{I}$, $n, -l$ is future-pointing null.*

Blocks $\bar{I}'$ and $\bar{I}\bar{I}'$ are defined to have the opposite time orientation with respect to their $\bar{I}$ and $\bar{I}\bar{I}$ counterparts. I.e.,

- *In block $\bar{I}'$, $-n, -l$ is future-pointing null.*
- *In block $\bar{I}\bar{I}'$, $-n, l$ is future-pointing null.*

In the remainder of this section, we show that the direction of future is continuously assigned.

Theorem 11. *In $\bar{I}$, along n, $\bar{t}$ is a constant, and t^* is increasing.*

Proof. In $\bar{K}$, since $n = \partial_{\bar{r}}$, we have that $\bar{t}$ is a constant along n. Since

$$t^* = \bar{t} - 2T(r)$$

we have that

$$\frac{dt^\star}{d\bar{r}} = -2\frac{(r^2 + a^2)}{\Delta} < 0 \ .$$

Therefore, $t^\star$ decreases along increasing $\bar{r}$. But n flows along decreasing values of $\bar{r}$. □

The following exercises check the result above for the remaining three blocks.

Problem 15. *In $\bar{I}$, along l, $t^\star$ is a constant, and $\bar{t}$ is increasing.*

Problem 16. *In $\bar{I}\bar{I}$, along n, $\bar{t}$ is a constant, and $t^\star$ is decreasing.*

Problem 17. *In $\bar{I}\bar{I}$, along l, $t^\star$ is a constant, and $\bar{t}$ is decreasing. Note, here it is $-l$ that is future pointing, so do not let Figure 7 confuse you.*

Problem 18. *In $\bar{I}'$, along n, $\bar{t}$ is a constant, and $t^\star$ is increasing.*

Problem 19. *In $\bar{I}'$, along l, $t^\star$ is a constant, and $\bar{t}$ is increasing.*

Problem 20. *In $\bar{I}\bar{I}'$, along n, $\bar{t}$ is a constant, and $t^\star$ is decreasing.*

And finally

Problem 21. *In $\bar{I}\bar{I}'$, along l, $t^\star$ is a constant, and $\bar{t}$ is decreasing.*

From theorem 11, definition 15, and the problems above, we see that $n, -n, l, -l$ are correctly drawn in figure 7. Furthermore, the assignment of future cones is continuous.

- Along $\bar{I}$ and $\bar{I}\bar{I}$, n is smooth.
- Along $\bar{I}'$ and $\bar{I}\bar{I}'$, $-n$ is smooth.
- Along $\bar{I}\bar{I}'$ and $\bar{I}$, l is smooth.
- Along $\bar{I}'$ and $\bar{I}\bar{I}$, $-l$ is smooth.

3.5 The Kerr Metric About The Extended Outer Horizon

We will now write the Kerr metric in the chart $(U^+, V^+, \theta^+, \varphi^+)$. It will turn out that the metric is valid when $U^+ = 0 = V^+$ and so we include this point as well.

Definition 16. *Following O'Neill [1]*

- *The manifold defined by*

  $$-\pi/2 < U^+, V^+ < \pi/2, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi^+ < 2\pi$$

  with be referred to as $D(r_+)$. Here φ^+ is understood to be a cyclic coordinate.
- *$U^+ = 0 = V^+$ is the crossing sphere of $D(r_+)$.*
- *The region defined by $\bar{t} \to -\infty, r^\star = r_+$ is the horizontal-long horizon. This includes the crossing sphere.*
- *The region defined by $t^\star \to \infty, \bar{r} = r_+$ is the vertical long horizon. This includes the crossing sphere.*

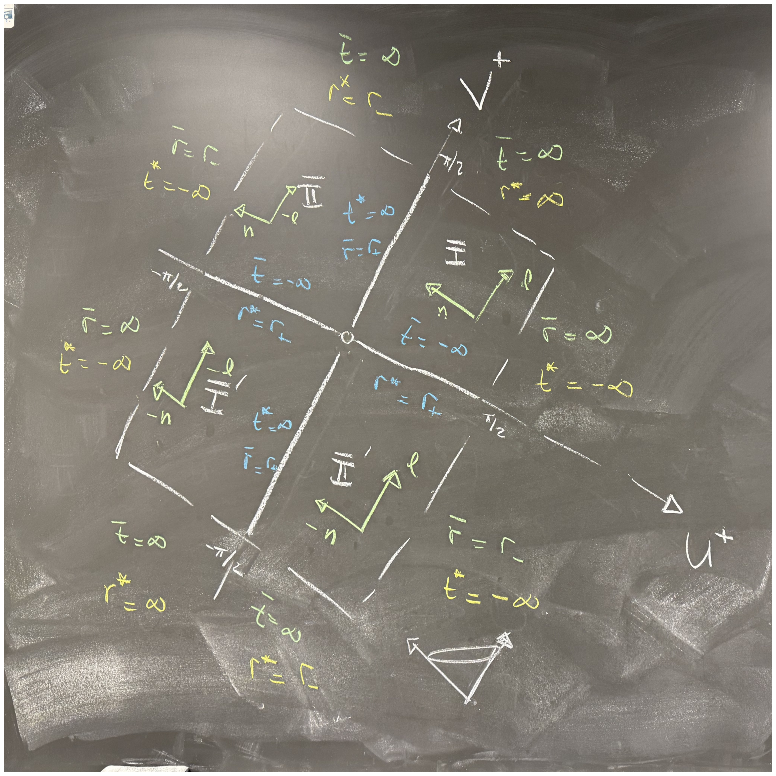

Figure 7: The Spacetime described by the chart $(U^+, V^+, \theta, \varphi^+)$.

- *The topology of $D(r_+)$ can be now fixed as*

$$(-\pi/2,\ \pi/2) \times (-\pi/2,\ \pi/2) \times \mathbb{S}^2(\theta^+, \varphi^+)$$

We will initially write the metric in the $(u^+, v^+, \theta^+, \varphi^+)$ coordinate system (see eq.(27) for the definitions of u^+ and v^+). First, a few preliminaries.

Theorem 12. *The radius function r is well defined on $D(r_+)$.*

Proof. Using the expressions for $\tan U^+$ and $\tan V^+$ in all four of patches, we see that

$$u^+v^+ = (r - r_+)\exp(2\kappa_+ r)(r - r_-)^{\kappa_+/\kappa_-}\ . \tag{29}$$

Since r is already defined everywhere except at the crossing sphere, we will show that the above equation can be inverted to solve for r near $r = r_+$. Let

$$f(r) = \exp(2\kappa_+ r)(r - r_-)^{\kappa_+/\kappa_-}\ .$$

Then

$$\frac{d}{dr}u^+v^+ = f(r) + (r - r_+)\frac{df}{dr}\ .$$

Therefore

$$\frac{d}{dr}u^+v^+|_{r_+} = f(r_+) \neq 0\ .$$

□

The next result stems from a direct computation.

Theorem 13. *In $D(r_+)$, in every Boyer-Lindquist block*

$$|u^+| = \exp(\kappa_+(T(r) - t)) \qquad |v^+| = \exp(\kappa_+(T(r) + t)) \tag{30}$$

and

$$\varphi^+ = \varphi - \frac{a}{r_+^2 + a^2} t \,. \tag{31}$$

Inverting the above equations we get

$$T(r) = \frac{1}{2\kappa_+} \ln |u^+ v^+| \qquad t = \frac{1}{2\kappa_+} \ln |v^+/u^+| \,, \tag{32}$$

and

$$\varphi = \varphi^+ + \frac{a}{r_+^2 + a^2} t \,. \tag{33}$$

Theorem 14. *In the Boyer-Lindquist blocks of $D(r_+)$*

$$dr = \frac{(r - r_-)\, G_+(r)}{2\kappa_+(r^2 + a^2)} (u^+ dv^+ + v^+ du^+) \,,$$

$$dt = \frac{G_+(r)}{2\kappa_+(r - r_+)} (u^+ dv^+ - v^+ du^+) \,,$$

and

$$d\varphi = d\varphi^+ + \frac{a\, G_+(r)}{(r - r_+)(r_+ - r_-)} (u^+ dv^+ - v^+ du^+) \,.$$

As expected

$$d\theta = d\theta^+ \,.$$

Here, the analytic function

$$G_+(r) = (r - r_+)/(u^+ v^+) = \exp(-2\kappa_+ r)(r - r_-)^{-\kappa_+/\kappa_-} \,.$$

Proof. From problem 18,

$$dr = \frac{\Delta}{r^2 + a^2}\, dT \,,$$

and from eq.(32) we get that

$$dT = \frac{1}{2\kappa_+ u^+ v^+} d(u^+ v^+) \,.$$

The above two equations give the required form of dr in the theorem. The remainder of the differentials are obtained in the same manner. □

Since the above Boyer-Lindquist differentials are dense in $D(r_+)$, a mere substitution of the results of the theorem above into eq.(1) leads to the metric on $D(r_+)$.

Theorem 15. *In $(u^+, v^+, \theta^+ = \theta, \varphi^+)$ coordinate system, the metric on $D(r_+)$ takes the form*

$$\begin{aligned}
&\frac{G_+^2 a^2 \sin^2\theta}{4\kappa_+^2 \rho^2} \frac{(r - r_-)(r + r_+)}{(r^2 + a^2)(r_+^2 + a^2)} \left[\frac{\rho^2}{r^2 + a^2} + \frac{\rho_+^2}{r_+^2 + a^2}\right] (u^{+2} dv^{+2} + v^{+2} du^{+2}) \\
&+ \frac{G_+(r - r_-)}{4\kappa_+^2 \rho^2} \left[\frac{\rho^4}{(r^2 + a^2)^2} + \frac{\rho_+^4}{(r_+^2 + a^2)^2}\right] (du^+ \otimes dv^+ + dv^+ \otimes du^+) \\
&+ \frac{G_+^2 a^2 \sin^2\theta}{4\kappa_+^2 \rho^2} \frac{(r + r_+)^2}{(r_+^2 + a^2)} (u^{+2} dv^{+2} + v^{+2} du^{+2} - u^+ v^+ du^+ \otimes dv^+ - u^+ v^+ dv^+ \otimes du^+) \\
&+ \frac{G_+ a \sin^2\theta}{\kappa_+ \rho^2 (r_+^2 + a^2)} \left[\rho_+^2 (r - r_-) + (r^2 + a^2)\,(r + r_+)\right] (u^+ dv^+ - v^+ du^+) d\varphi^+ \\
&\rho^2 d\theta^2 + g_{\varphi\varphi}\, d\varphi^{+2} \,.
\end{aligned} \tag{34}$$

Proof. This is nothing more than a simple calculation that is presented in [1].[5] □

To go from $(u^+, v^+, \theta^+ = \theta, \varphi^+)$ to $(U^+, V^+, \theta^+ = \theta, \varphi^+)$, all we need is the substitution

$$du^+ = \sec^2 U^+ dU^+ \quad \text{and} \quad dv^+ = \sec^2 V^+ dV^+ \ .$$

Lemma 4. *The metric given by eq.(34) is non-degenerate on the crossing sphere, and is extendable to the poles.*

Proof. Extendibility to the poles is implied by the coordinates T, X, Y, Z. On the crossing sphere, $u^+ = 0 = v^+$, and the metric reduces to

$$\frac{G_+(r_+ - r_-)\rho_+^2}{2\kappa_+^2(r_+^2 + a^2)^2}\,(du^+ \otimes dv^+ + dv^+ \otimes du^+) + \rho^2 d\theta^2 + g_{\varphi\varphi}\, d\varphi^{+2} \ , \tag{35}$$

and this is clearly non-degenerate. □

Theorem 16. *The Boyer-Lindquist coordinate vector fields are given by*

$$\partial_t = \kappa_+\big[- u^+\partial_{u^+} + v^+\partial_{v^+}\big] - \frac{a}{r_+^2 + a^2}\,\partial_{\varphi^+} \ ,$$

$$\partial_r = \kappa_+ \frac{r^2 + a^2}{\Delta}\big[u^+\partial_{u^+} + v^+\partial_{v^+}\big] \ ,$$

$$\partial_\theta = \partial_{\theta^+} \ ,$$

and

$$\partial_\varphi = \partial_{\varphi^+} \ ,$$

of course, on the axis $\partial_{\varphi^+} = 0$.

Proof. Clearly,

$$\partial_t = \partial_t(u^+)\,\partial_{u^+} + \partial_t(v^+)\,\partial_{v^+} + \ \partial_t(\varphi^+)\,\partial_{\varphi^+} \ .$$

A simple substitution and differentiation gives the necessary result. □

3.6 The Long Horizon(s)

The principal null geodesics that live on the r_+ horizon never leaves $D(r_+)$. Consequently, we must make sure that, as such, they are complete. Consider the long vertical horizon in $D(r_+)$. Excluding the crossing sphere, this is the disjoint line $U^+ = 0\ , V^+ > 0$, and $U^+ = 0\ , V^+ < 0$. These future pointing pre-geodesics are integral curves of

$$V_+ \equiv (r_+^2 + a^2)\partial_{\bar t} + a\partial_{\bar\varphi}$$

when $V^+ > 0$ (problem 9) and

$$V'_+ \equiv -[(r_+^2 + a^2)\partial_{\bar t} + a\partial_{\bar\varphi}]$$

when $V^+ < 0$ because of the reverse time orientation.

Theorem 17. *In coordinates* $(\bar t,\ \bar r,\ \bar\theta,\ \bar\varphi)$,

$$((r_+^2 + a^2)f(s),\ r_+,\ \theta_0,\ af(s) + \beta_0)$$

is the integral curve of

$$V_+ = (r_+^2 + a^2)\partial_{\bar t} + a\partial_{\bar\varphi} \ .$$

s is an affine parameter when $f(s) = \alpha_0^{-1}\ln(\alpha_0\ s) + C$ for $0 < s < \infty$ [6]*, where $\alpha_0 = (r_+ - M)$. Here, C and β_0 are integration constants.*

[5] However, there maybe a sign error in O'Neills version: there is a negative sign in the fourth line of the above equation.

[6] If you are worried about $\lim_{s\to 0}\bar t(s) \to -\infty$; don't be. That is exactly the case in the long vertical horizon near the crossing sphere. Apparently $\bar\varphi(s)$ behaves the same way.

Proof. Let

$$\alpha(s) \equiv ((r_+^2 + a^2) f(s),\ r_+,\ \theta_0,\ a f(s)) \ .$$

Clearly,

$$\dot\alpha = \dot f\ (r_+^2 + a^2,\ 0,\ 0,\ a) = \dot f\ V_+ \ ,$$

where $\cdot$ is the derivative with respect to s. Then,

$$\nabla_{\dot\alpha}\ \dot\alpha = \dot f^2 \nabla_{V_+} V_+ + [\dot f\ V_+(\dot f)]\ V_+ \ .$$

Also, we have from eq.(12) that

$$\nabla_{V_+} V_+ = (r_+ - M)\ V_+ = \alpha_0\ V_+ \ . \qquad (36)$$

But, $\dot f$ when viewed as a function on the curve $\alpha(s)$,

$$\dot f(s) = \dot f(\bar t(s), \bar\varphi(s)) \ ,$$

and so

$$\begin{aligned} \dot f\ V_+(\dot f) &= \dot f \left((r_+^2 + a^2) \frac{\partial}{\partial \bar t} + a \frac{\partial}{\partial \bar\varphi} \right) \dot f \\ &= (r_+^2 + a^2) \dot f \frac{\partial \dot f}{\partial \bar t} + a \dot f \frac{\partial \dot f}{\partial \bar\varphi} \\ &= \frac{d\bar t}{ds}\ \partial_{\bar t} \dot f + \frac{d\bar\varphi}{ds}\ \partial_{\bar\varphi} \dot f = \ddot f \ . \end{aligned}$$

Therefore, if s is to be an affine parameter, we must have

$$\nabla_{\dot\alpha}\ \dot\alpha = [\ddot f + \alpha_0\ \dot f^2]\ V_+ = 0 \ ,$$

i.e.,

$$\ddot f + \alpha_0\ \dot f^2 = 0 \ .$$

This has the unique solution

$$f(s) = \alpha_0^{-1} \ln |\alpha_0\ s + C_1| + C \ ,$$

here C and C_1 are integration constants. Setting $C_1 = 0$ amounts to fixing the parameter lower limit to be $s \to 0$. □

In exactly the same way we get the following result.

Theorem 18. *In coordinates* $(\bar t,\ \bar r,\ \bar\theta,\ \bar\varphi)$*,*

$$((r_+^2 + a^2) f(s),\ r_+,\ \theta_0,\ a f(s) + \beta_0') \ .$$

is the integral curve of

$$V_+' = -[(r_+^2 + a^2)\partial_{\bar t} + a \partial_{\bar\varphi}] \ .$$

s is an affine parameter when $f(s) = \alpha_0^{-1} \ln(-\alpha_0\ s) + c$ for $-\infty < s < 0$, where $\alpha_0 = (r_+ - M)$. Here, c and β_0' are integration constants.

We now paste the two intergral curves together using coordinates $(U^+, V^+, \theta^+, \varphi^+)$.

Theorem 19. *The null geodesic generator in the long vertical horizon of $D(r_+)$ is complete* [7] *, and its integral curve is given by*

$$U^+ = 0 \ , \quad V^+ = \arctan s \ , \quad \theta = \text{const} \ , \quad \text{and} \quad \varphi^+ = \text{const}, \quad \forall s \in R \ .$$

[7] The affine parameter s of the geodesics runs from $-\infty$ to ∞.

Proof. Since $r = r_+$, clearly $U^+ = 0$, and from theorem (17) we see that

$$V^+ = \arctan\exp(\kappa_+ \bar{t}) = \arctan\exp[\frac{\kappa_+(r_+^2 + a^2)}{\alpha_0} \ \ln(\beta\ s)\ ]\ .$$

Here β is a new constant defined by $C = 1/\alpha_o \ln(\beta/\alpha_o)$, where C is the constant in theorem (17). Note

$$\frac{\kappa_+(r_+^2 + a^2)}{\alpha_0} = \frac{\kappa_+(r_+^2 + a^2)}{(r_+ - r_-)/2} = 1\ .$$

Therefore,

$$V^+ = \arctan(\beta\ s)$$

when $0 < s < \infty$. Here

$$\varphi^+ = \bar{\varphi} - \frac{a}{r_+^2 + a^2}\bar{t}\ - \left[A(r) - \frac{a}{r_+^2 + a^2}T(r)\right]\ .$$

Substituting $\bar{\varphi} = af(s) + \beta_0$ and $\bar{t} = (r_+^2 + a^2)f(s)$ from theorem (17), we see that

$$\varphi^+ = \beta_0^+ - \left[A(r_+) - \frac{a}{r_+^2 + a^2}T(r_+)\right]$$

which is finite from corollary (13). Clearly φ^+ is a constant. The exact same is true for the $V^+ < 0$ portion, except that here $-\infty < s < 0$. We can glue two such curves at the limiting point $s \to 0$ when θ_0 and φ^+ are the same. This geodesic curve is the complete. Incidently a change in the affine parameter given by $\beta\ s \to s$ gives the necessary result. □

3.7 The Maximal Submanifold Around The Inner Horizon

In exactly the same manner as we defined $D(r_+)$, we will include the $r = r_-$ horizon in the submanifold $D(r_-)$.

Definition 17. *Define real valued functions U^- and V^- by:*

On $\bar{I}\bar{I}$, $\tan U^- = \exp(-\kappa_-\ t^\star)\ ,\ \tan V^- = -\exp(\kappa_-\ \bar{t})\ ,$

On $\bar{I}\bar{I}\bar{I}$, $\tan U^- = -\exp(-\kappa_-\ t^\star)\ ,\ \tan V^- = -\exp(\kappa_-\ \bar{t})\ ,$

and

On $\bar{I}\bar{I}'$, $\tan U^- = -\exp(-\kappa_-\ t^\star)\ ,\ \tan V^- = \exp(\kappa_-\ \bar{t})\ ,$

On $\bar{I}\bar{I}\bar{I}'$, $\tan U^- = \exp(-\kappa_-\ t^\star)\ ,\ \tan V^- = \exp(\kappa_-\ \bar{t})\ ,$

$$\varphi^- = \frac{1}{2}\left[\bar{\varphi} + \varphi^\star - \frac{a}{r_-^2 + a^2}(\bar{t} + t^\star)\right]\ ,$$

and $\theta^- = \theta$. These function are defined only on the Boyer-Lindquist blocks (which do not include the horizons). As before, we will set $u^- = \tan U^-$ and $v^- = \tan V^-$.

In exactly the same way as in the case of $D(r_+)$,

Theorem 20. *The functions $(U^-, V^-, \theta^-, \varphi^-)$ can be analytically extended to r_- in $D(r_-)$.*

Problem 22. *Complete figure 8 for $D(r_-)$ (I have not labelled $t^\star = \pm\infty$ in blocks $\bar{I}\bar{I}'$ and $\bar{I}\bar{I}\bar{I}'$). Make sure to point out future-pointing pairs of $n, l, -n, -l$ in each of the four blocks. Argue why your choice of time orientation is smooth in $D(r_-)$.*

Problem 23. *In $D(r_-)$, verify that the long horizon contain complete geodesic generators.*

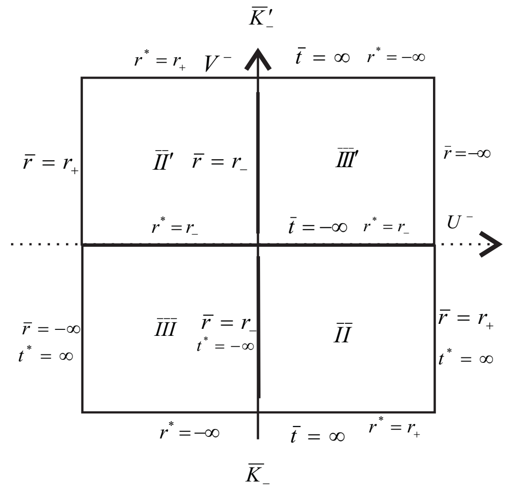

Figure 8: Penrose diagram for the spacetime submanifold $D(r_-)$. $r_- \in D(r_-)$. The boundary of the rectangle is not included in the open submanifold $D(r_-)$.

3.8 The Kerr Metric about the Extended Inner Horizon

The results below are the $D(r_-)$ analogue of the corresponding theorems in section 3.5, so we state them without proof.

Theorem 21. *In $D(r_-)$, in every Boyer-Lindquist block*

$$|u^-| = \exp(\kappa_-(T(r) - t)) \qquad |v^-| = \exp(\kappa_-(T(r) + t)) \tag{37}$$

and

$$\varphi^- = \varphi - \frac{a}{r_-^2 + a^2} t \, . \tag{38}$$

Inverting the above equations we get

$$T(r) = \frac{1}{2\kappa_-} \ln |u^- v^-| \qquad t = \frac{1}{2\kappa_-} \ln |v^-/u^-| \, , \tag{39}$$

and

$$\varphi = \varphi^- + \frac{a}{r_-^2 + a^2} t \, . \tag{40}$$

Theorem 22. *In the Boyer-Lindquist blocks of $D(r_-)$*

$$dr = \frac{(r - r_+)\, G_-(r)}{2\kappa_-(r^2 + a^2)} (u^- dv^- + v^- du^-) \, ,$$

$$dt = \frac{G_-(r)}{2\kappa_-(r - r_-)} (u^- dv^- - v^- du^-) \, ,$$

and

$$d\varphi = d\varphi^- + \frac{a\, G_-(r)}{(r - r_-)(r_- - r_+)} (u^- dv^- - v^- du^-) \, .$$

As expected

$$d\theta = d\theta^+ \, .$$

Here, the analytic function

$$G_-(r) = (r - r_-)/(u^- v^-) = -\exp(-2\kappa_- r)(r_+ - r)^{-\kappa_-/\kappa_+} \, .$$

Theorem 23. *In the $(u^-, v^-, \theta^- = \theta, \varphi^-)$ coordinate system, the metric on $D(r_-)$ takes the form*

$$\frac{G_-^2 a^2 \sin^2\theta}{4\kappa_-^2 \rho^2} \frac{(r-r_+)(r+r_-)}{(r^2+a^2)(r_-^2+a^2)} \left[\frac{\rho^2}{r^2+a^2} + \frac{\rho_-^2}{r_-^2+a^2}\right] (u^{-2}dv^{-2} + v^{-2}du^{-2})$$
$$+\frac{G_-(r-r_+)}{4\kappa_-^2\rho^2} \left[\frac{\rho^4}{(r^2+a^2)^2} + \frac{\rho_-^4}{(r_-^2+a^2)^2}\right] (du^- \otimes dv^- + dv^- \otimes du^-)$$
$$+\frac{G_-^2 a^2 \sin^2\theta}{4\kappa_-^2\rho^2} \frac{(r+r_-)^2}{(r_-^2+a^2)} (u^{-2}dv^{-2} + v^{-2}du^{-2} - u^-v^-du^- \otimes dv^- - u^-v^-dv^- \otimes du^-)$$
$$+\frac{G_- a \sin^2\theta}{\kappa_-\rho^2(r_-^2+a^2)} \left[\rho_-^2(r-r_+) + (r^2+a^2)\,(r+r_-)\right] (u^-dv^- - v^-du^-) \times d\varphi^-$$
$$+\rho^2 d\theta^2 + g_{\varphi\varphi}\, d\varphi^{-2}\ . \tag{41}$$

Proof. This is nothing more than a simple calculation that is presented in [1].[8] □

Once again, we see that the metric is not singular at the crossing sphere of $D(r_-)$. To go from $(u^-, v^-, \theta^- = \theta, \varphi^-)$ to $(U^-, V^-, \theta^- = \theta, \varphi^-)$, all we need is the substitution

$$du^- = \sec^2 U^- dU^- \quad \text{and} \quad dv^- = \sec^2 V^- dV^-\ .$$

Lemma 5. *The metric in eq.(41) is non-degenerate on the crossing sphere, and is extendable to the poles.*

Proof. On the crossing sphere, $u^- = 0 = v^-$, and the metric reduces to

$$\frac{G_-(r_- - r_+)\rho_-^2}{2\kappa_-^2(r_-^2+a^2)^2} (du^- \otimes dv^- + dv^- \otimes du^-) + \rho^2 d\theta^2 + g_{\varphi\varphi}\, d\varphi^{-2}\ , \tag{42}$$

and this is clearly non-degenerate. □

Theorem 24. *The Boyer-Lindquist coordinate vector fields are given by*

$$\partial_t = \kappa_-\big[-u^-\partial_{u^-} + v^-\partial_{v^-}\big] - \frac{a}{r_-^2+a^2}\,\partial_{\varphi^-}\ ,$$

$$\partial_r = \kappa_- \frac{r^2+a^2}{\Delta}\big[u^-\partial_{u^-} + v^-\partial_{v^-}\big]\ ,$$
$$\partial_\theta = \partial_{\theta^-}\ ,$$

and

$$\partial_\varphi = \partial_{\varphi^-}\ ,$$

of course, on the axis $\partial_{\varphi^-} = 0$.

3.9 Ad Infinitum ...

It should be clear by now as to how we can stack an infinite array of alternating $D(r_\pm)$ blocks to form the maximal Kerr geometry. The resulting Penrose diagram is schematically represented in figure (9). By construction, the principal null geodesics are complete (including the null generators of horizons). This is the maximal slow-Kerr geometry.

[8] However, there maybe a sign error in O'Neills version: there is a negative sign in the fourth line of the above equation.

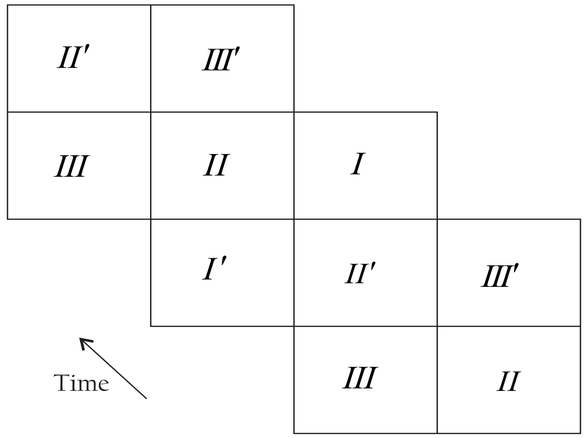

Figure 9: Penrose diagram representing the maximal Kerr geometry. Here, the central $D(r_+)$ is sandwiched between two $D(r_-)$. In the complete maximal extension, such alternating open submanifolds continue ad infinitum.

References

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