North Carolina Journal of
Mathematics and Statistics
Volume 5, Pages 28–40 (Accepted May 9, 2019, published May 10, 2019)
ISSN 2380-7539

Implementing the Lifetime Performance Index of Products with a Two-Parameter Rayleigh Distribution Under a Progressively type II Right Censored Sample

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ABSTRACT. In manufacturing, quality control is a process that ensures customers receive products free from defects and meet their needs. Process capability analysis has been widely applied in the field of quality control to find out how well a given process meets a set of specification limits. The lifetime performance index C_L, a type of process capability index is used to measure the larger-the-better type quality characteristics. Under the assumption of Two-Parameter Rayleigh Distribution, this study constructs a maximum likelihood estimator of C_L based on the progressively type II right censored sample. The maximum likelihood estimator of C_L is then utilized to develop the new hypothesis testing procedure. The testing procedure can be employed the testing procedure to determine whether the lifetime of a product adheres to the required level.

1. Introduction

Process capability indices are widely used in the manufacturing industry to measure process potential and performance. Process capability indices are utilized to evaluate whether product quality meets required performance level and customer expectations. Since the lifetime of electronic components exhibit larger-the-better quality characteristics of time orientation, Montgomery (2005) and Kane (1986) proposed a process capability index C_L (or C_{PL}) for evaluating the lifetime performance of electronic components, where L is the lower specification limit. However, the normality assumption the often not valid, is common in process capability analysis.

There have been numerous works on statistical inference for the lifetime performance index based on usual type-II and progressive type-II censoring schemes with various lifetime distributions. Classical statistical inference for a lifetime performance index based on one and two parameter exponential lifetimes have been discussed by many authors. Under the assumption of one-parameter exponential distribution, Tong et al. (2002) developed a uniformity minimum variance unbiased estimator (UMVUE) of, and testing procedure for a lifetime performance index based on a complete sample. As an extension of this work, Tong et al. (2002) developed the UMVUE, confidence intervals of and testing procedure for the lifetime performance index based on a type-II right censored sample. Under the assumption of exponential distribution, Lee et al. (2009) constructed a maximum likelihood estimator (MLE) of lifetime performance index based on a progressively type-II right censored sample. They then utilized the MLE of a lifetime performance index to develop a hypothesis testing procedure in the condition of a known Lee et al. (2011b) constructed a

Received by the editors April 30, 2019.
2010 *Mathematics Subject Classification.* 47N30.
Key words and phrases. Lifetime data; lifetime performance index; lower specification limit; progressive type II censored sample.

Bayes estimator under a one-parameter Rayleigh distribution with a progressive type-II right censored sample. The Bayes estimator of lifetime performance index was then utilized to develop a credible interval in the condition of a known L. For more examples, some references are Dey et al. (2016), Lee et al. (2010), Hong et al. (2009), Lee et al. (2009), Lee et al. (2011a), and Gunasekera and Wijekularathna (2018).

Motivated by these works, in this article, under a two-parameter Rayleigh distributed lifetime, we implemented a maximum likelihood estimator of a lifetime performance index based on a type II right-censored sample. The MLE of a lifetime performance index was then utilized to develop a new hypothesis testing procedure in the conditions of a known L.

In life testing experiments, it can be difficult for experimenters to observe the lifetimes of all products tested due to time or other resource restrictions. Therefore, censored samples are commonly used in practice. In this study, we consider a progressive type-II censoring scheme in which we only observe the failure times. The m ordered observed failure times are denoted by $X_{1,n} \leq X_{2,n} \leq ..., \leq X_{m,n}$, and the number of surviving units removed at each failure time stage is denoted by $R_1, R_2, ..., R_m$. It is clear that $R_m = n - \sum_{j=1}^{m-1} R_j - m$ and $0 \leq R_i \leq n - \sum_{j=1}^{i-1} R_j - i$ for $i = 2, 3, ..., m-1$.

The rest of this paper is organized as follows: In section 2 some properties of a lifetime performance index for the lifetime of the product based on a progressively type II censored sample are discussed. In section 3, we investigate the relationship between a lifetime performance index and the conforming rate of products. An MLE of the lifetime performance index is proposed in section 4.

2. The Lifetime Performance Index

Suppose that the lifetime of products, X, has a two-parameter Rayleigh distribution. X will then have the following probability density function (p.d.f):

$$f(x, \lambda, \mu) = 2\lambda(x-\mu)e^{-\lambda(x-\mu)^2} \; ; \; x > \mu, \; \mu > 0 \tag{2.1}$$

and a cumulative distribution function (c.d.f):

$$F(x, \lambda, \mu) = 1 - e^{-\lambda(x-\mu)^2} \tag{2.2}$$

where λ and μ are scale and location parameters respectively.

Two-parameter Rayleigh distribution can be converted to a one parameter Rayleigh distribution, by using the transformation $Y = \sqrt{2}\lambda^{\frac{3}{2}}(x-\mu)$ which has a p.d.f and c.d.f.

$$f_Y(y|\lambda) = \frac{y}{\lambda^2} e^{-\frac{y^2}{2\lambda^2}} \; , \; y > 0, \lambda > 0 \tag{2.3}$$

and

$$F_Y(y|\lambda) = 1 - e^{-\frac{y^2}{2\lambda^2}} \; , \; y > 0, \lambda > 0 \tag{2.4}$$

respectively. Clearly, a longer lifetime implies a better product quality. Hence, in this case the lifetime reflects larger-the-better type quality characteristics. The lifetime is generally required to exceed the lower specification in L years to be economically profitable to investors.

Montgomery (2005) developed a capability index C_L to measure the larger-the-better quality characteristic. C_L as defined as follows:

$$C_L = \frac{\mu - L}{\sigma} \tag{2.5}$$

where the process mean μ, the process standard deviation σ, and the lower specification limit is L. C_L can be defined as the lifetime performance index to assess the performance of business systems.

The lifetime performance index C_{L_Y} can be written as:

$$C_{L_Y} = \frac{\mu_y - L_Y}{\sigma_Y} = \sqrt{\frac{\pi}{4-\pi}} - \sqrt{\frac{2}{4-\pi}}\frac{L_Y}{\lambda}, \qquad \infty < C_{L_Y} < \sqrt{\frac{\pi}{4-\pi}} \tag{2.6}$$

where the process mean is $\mu_Y = \sqrt{\frac{\pi}{2}}\lambda$ the process standard deviation is $\sigma_Y = \sqrt{\frac{4-\pi}{2}}\lambda$ and the L_Y is the lower specification limit of Y.

The failure rate function $r(y)$ is defined by (see Lee et al., 2009):

$$r(y) = \frac{f_Y(y|\lambda)}{(1 - F_Y(y|\lambda))} = \frac{y}{\lambda^2}\ ;\ y > 0, \lambda > 0 \tag{2.7}$$

The data transformation $Y = \sqrt{2}\lambda^{\frac{3}{2}}(x - \mu)\ ;\ x > 0, x > \mu$ is one-to-one and strictly increasing. Therefore, data set of X and the transformed data set of Y have the same effect in assessing the lifetime performance of products.

By (2.6)-(2.7), it can be seen that the larger the λ, the smaller the failure rate and the larger the lifetime performance index C_{L_Y}. Therefore, the lifetime performance index C_{L_Y} reasonably and accurately represents the lifetime performance of products.

3. The Conforming Rate

The product is called conforming if the lifetime of the product exceeds the lower specification limit L_Y. The ratio of conforming products is known as the conforming rate P_r which can be defined as (see Lee et al., 2011b):

$$P_r = P(Y \geq L_Y) = e^{-\frac{1}{2}\left(\sqrt{\frac{\pi}{2}} - \sqrt{\frac{4-\pi}{2}}C_{L_Y}\right)^2};\ -\infty < C_{L_Y} < \sqrt{\frac{\pi}{4-\pi}} \tag{3.1}$$

It is obvious that a strictly positive relationship exists between the conforming rate P_r and the lifetime performance index C_{L_Y}. The conforming rate can be estimated by dividing the number of conforming products by the total number of products sampled.

Montgomery (2005) suggested that the sample size must be large to accurately estimate the conforming rate. However, a large sample size is usually not practical from the perspective of cost, since collecting the lifetime data of new products involves damaging the products which may prove cost prohibitive.

A complete sample is also impractical due to several reasons such as time limitation, lack of funds, and mechanical difficulties. Since a one-to-one mathematical relationship exists between the conforming rate P_r and C_{L_Y}, the lifetime performance index will be flexible and can be an effective tool for estimating the conforming rate, P_r.

4. MLE Unbiased Estimator of C_{L_Y}

As mentioned above, due to several reasons such as lack of materials resources, lack of funds, mechanical or experiment difficulties, the experimenter may not always be in a position to observe the lifetimes of all test products. Therefore, use of censored samples is recommended. In this study, we consider the case of two parameter Rayleigh distribution under progressively type II right censored data.

Consider that $Y_{1:m:n} \leq Y_{2:m:n} \leq ... \leq Y_{m:m:n}$ is the corresponding progressively type II right censored sample, with a censoring scheme $R = (R_1, R_2, ..., R_m)$. The joint p.d.f of $Y_{1:m:n}, Y_{2:m:n}, ..., Y_{i:m:n}$ is given by (see Balakrishnan and Aggarwala, 2000)

$$L(y, \lambda) = c \prod_{i=1}^{m} f_Y\left(y_{i:m:n}\right)\left[1 - F_Y(y_{i:m:n})\right]^{R_i} \tag{4.1}$$

where

$$c = n \prod_{i=1}^{m-1} \left(n - \sum_{j=1}^{i} (R_j + 1)\right), \tag{4.2}$$

and $f_Y(y_{i:m:n})$ and $F_Y(y_{i:m:n})$ are the p.d.f and c.d.f of Y as in (2.3) and (2.4), respectively.

So, the likelihood function for $Y_{1:m:n}, Y_{2:m:n}, ..., Y_{i:m:n}$ is given by

$$L(\lambda) = c\lambda^{-2m} \left(\prod_{i=1}^{m} y_{i:m:n}\right) e^{\frac{1}{2\lambda^2} \sum_{i=1}^{m} (R_i+1) Y_{i:m:n}^2} \tag{4.3}$$

where c is given by (4.2).

It is easy to obtain the MLE of λ given by

$$\hat{\lambda}_{MLE} = \left[\frac{1}{2m} \sum_{i=1}^{m} (R_i + 1) Y_{i:m:n}^2\right]^{\frac{1}{2}}$$

where R_i and m given in the above definition.

By the in-variance property of the MLE (Zehna, 1966), the MLE $\hat{C}_{L_Y}$ of C_{L_Y} can be written as:

$$\hat{C}_{L_Y} = \sqrt{\frac{\pi}{4 - \pi}} - \sqrt{\frac{2}{4 - \pi}} \frac{L_Y}{\hat{\lambda}_{MLE}} \tag{4.4}$$

Moreover, we can also show that $\frac{W}{\lambda^2} \sim \chi^2_{(2m)}$, where $W = \sum_{i=1}^{m} (R_i + 1) Y_{i:m:n}^2$.

Hence, the expectation of $\hat{C}_{L_Y}$ can be derived as follows:

$$E(\hat{C}_{L_Y}) = \sqrt{\frac{\pi}{4 - \pi}} - \sqrt{\frac{2}{4 - \pi}} L_Y E\left(\frac{1}{\sqrt{\frac{W}{2m}}}\right) \tag{4.5}$$

$$\Rightarrow E\left(\frac{1}{\sqrt{\frac{W}{2m}}}\right) = \sqrt{\frac{2m}{\lambda^2}} E\left(\left(\frac{W}{\lambda^2}\right)^{-\frac{1}{2}}\right)$$
$$= \sqrt{\frac{2m}{\lambda^2}}\ \frac{2^{-\frac{1}{2}}\Gamma(m-\frac{1}{2})}{\Gamma(m)}$$
$$= \sqrt{\frac{m}{\lambda^2}}\ \frac{\Gamma(m-\frac{1}{2})}{\Gamma(m)}$$

$$\Rightarrow E(\hat{C}_{L_Y}) = \sqrt{\frac{\pi}{4-\pi}} - \sqrt{\frac{2}{4-\pi}} L_Y \sqrt{\frac{m}{\lambda^2}} \frac{\Gamma(m-\frac{1}{2})}{\Gamma(m)}$$

Since $E(\hat{C}_{L_Y}) \neq C_{L_Y}$, where $C_{L_Y} = \sqrt{\frac{\pi}{4-\pi}} - \sqrt{\frac{2}{4-\pi}} \frac{L_Y}{\lambda}$, the MLE $\hat{C}_{L_Y}$ is not an unbiased estimator of C_{L_Y}. $\hat{C}_{L_Y}$ can be modified as below:

$$\hat{C}'_{L_Y} = \sqrt{\frac{\pi}{4-\pi}} - \sqrt{\frac{2}{4-\pi}}.L_Y.\left(\sqrt{\frac{W}{2}}.\frac{\Gamma(m-\frac{1}{2})}{\Gamma(m)}\right)^{-1} \tag{4.6}$$

Now, $\hat{C}'_{L_Y}$ is an unbiased estimator of C_{L_Y}.

5. Testing Procedure for the Lifetime Performance Index

In this section, a statistical testing procedure to assess whether the lifetime performance index adheres to the required level will be constructed. Assuming that the required lifetime performance index value, C_{L_Y} is larger than C^*, where C^* denotes the target value, the null and alternative hypothesis can be represented as follows:

$$H_0\text{: the process is unreliable vs } H_a\text{: the process is reliable}$$

i.e.

$$H_0 : C_{L_Y} \leq C^* \quad vs \quad H_a : C_{L_Y} > C^*$$

The unbiased estimator $\hat{C}'_L$ of C_{L_Y} is used as the test statistic, so the rejection region can be expressed as $[\hat{C}'_L | \hat{C}'_L > C_0]$, for the given specified significance level α and the critical value C_0. Then the critical value C_0 can be calculated as follows:

$$
\begin{aligned}
&P(\hat{C}'_{L_Y} > C_0 | C_{L_Y} \le C^*) \le \alpha \\
&\quad \Rightarrow P\left(\sqrt{\frac{\pi}{4-\pi}} - \sqrt{\frac{2}{4-\pi}} L_Y \left(\sqrt{\frac{W}{2}}\frac{\Gamma(m-\frac{1}{2})}{\Gamma(m)}\right)^{-1} > C_0 \,\middle|\, C_{L_Y} \le C^*\right) \le \alpha \\
&\quad \Rightarrow P\left(\frac{1}{\left(\sqrt{\frac{\pi}{4-\pi}} - C_0\right)}\sqrt{\frac{2}{4-\pi}}\sqrt{2}L_Y\frac{\Gamma(m)}{\Gamma(m-\frac{1}{2})} < \sqrt{W} \,\middle|\, C_{L_Y} \le C^*\right) \le \alpha \\
&\quad \Rightarrow P\left(\frac{1}{\left(\sqrt{\frac{\pi}{4-\pi}} - C_0\right)}\sqrt{\frac{2}{4-\pi}}\sqrt{2}L_Y\frac{\Gamma(m)}{\Gamma(m-\frac{1}{2})}\frac{1}{\lambda} < \sqrt{\frac{W}{\lambda^2}} \,\middle|\, \sqrt{\frac{\pi}{4-\pi}} - \sqrt{\frac{2}{4-\pi}}\frac{L_Y}{\lambda} \le C^*\right) \le \alpha \\
&\quad \Rightarrow P\left(\frac{1}{\left(\sqrt{\frac{\pi}{4-\pi}} - C_0\right)}\sqrt{\frac{2}{4-\pi}}\sqrt{2}L_Y\frac{\Gamma(m)}{\Gamma(m-\frac{1}{2})}\frac{1}{\lambda} < \sqrt{\frac{W}{\lambda^2}} \,\middle|\, \sqrt{\frac{\pi}{4-\pi}} - \sqrt{\frac{2}{4-\pi}}\frac{L_Y}{\lambda} = C^*\right) = \alpha \\
&\quad \Rightarrow P\left(\frac{1}{\left(\sqrt{\frac{\pi}{4-\pi}} - C_0\right)}\sqrt{\frac{2}{4-\pi}}\sqrt{2}L_Y\frac{\Gamma(m)}{\Gamma(m-\frac{1}{2})}\frac{1}{\lambda} < \sqrt{\frac{W}{\lambda^2}} \,\middle|\, \lambda = \sqrt{\frac{2}{4-\pi}}\frac{L_Y}{\left(\sqrt{\frac{\pi}{4-\pi}} - C^*\right)}\right) = \alpha
\end{aligned}
$$

$$
\Rightarrow P\left(\frac{W}{\lambda^2} \le \left(\frac{\sqrt{2}\left(\frac{\pi}{4-\pi}\right) - C^*)}{\left(\sqrt{\frac{\pi}{4-\pi}} - C_0\right)}\frac{\Gamma(m)}{\Gamma(m-\frac{1}{2})}\right)^2\right) = 1 - \alpha \tag{5.1}
$$

where $\frac{W}{\lambda^2} \sim \chi^2_{2m}$.

From (5.1), utilizing function $CHIINV(1-\alpha, 2m)$ which represents the lower $100(1-\alpha)^{th}$ percentile of χ^2_{2m}.

$$
\left(\frac{\sqrt{2}\Gamma(m)(\sqrt{\frac{\pi}{4-\pi}} - C^*)}{\Gamma(m-\frac{1}{2})\left(\sqrt{\frac{\pi}{4-\pi}} - C_0\right)}\right)^2 = CHIINV(1-\alpha, 2m)
$$

is obtained. Thus, the critical value C_0 can be derived as:

$$
C_0 = \sqrt{\frac{\pi}{4-\pi}} - \frac{\Gamma(m)}{\Gamma(m-\frac{1}{2})}\left(\frac{2}{CHIINV(1-\alpha, 2m)}\right)^{-\frac{1}{2}}\left(\sqrt{\frac{\pi}{4-\pi}} - C^*\right) \tag{5.2}
$$

where C^*, α and m denote the target value, the specified significance level, and the number of observed failures before termination respectively. Moreover, we also find that C_0 is independent of n and $R_i,\ \ i = 1, 2, ..., m$. Table 5.1 and 5.2 list the critical values C_0 for k = $1(1)50$ (i.e. 1,2,3,...50) and $C = 0.1(0.1)0.9$ (i.e. 0.1, 0.2, 0.3,...0.9) at α = 0.01 and α = 0.05 respectively.

In addition, the level $(1-\alpha)$ one-sided confidence interval for C_L can be derived as follows: Since $\frac{W}{\lambda^2} \sim \chi^2_{2m}$ and $CHIINV(1-\alpha, 2m)$ which represents the lower $1-\alpha$ of χ^2_{2m}, we can derive the lower bound for C_{L_Y} as follows:

$$P\left(\frac{W}{\lambda^2} \le CHIINV(1-\alpha, 2m)\right) = 1-\alpha$$

$$\Rightarrow P\left(\frac{1}{\lambda} \le \left(\frac{CHIINV(1-\alpha, 2m)}{W}\right)^{\frac{1}{2}}\right) = 1-\alpha$$

$$\Rightarrow P\left(\sqrt{\frac{\pi}{4-\pi}} - \sqrt{\frac{2}{4-\pi}}\frac{L_Y}{\lambda} \ge \sqrt{\frac{\pi}{4-\pi}} - \sqrt{\frac{2}{4-\pi}}L_Y\left(\frac{CHIINV(1-\alpha, 2m)}{W}\right)^{\frac{1}{2}}\right) = 1-\alpha$$

$$\Rightarrow P\left(C_{L_Y} \ge \sqrt{\frac{\pi}{4-\pi}} - \sqrt{\frac{2}{4-\pi}}L_Y\left(\frac{CHIINV(1-\alpha, 2m)}{W}\right)^{\frac{1}{2}}\right) = 1-\alpha$$

But, from (4.6)

$$\frac{1}{\sqrt{W}} = \left(\sqrt{\frac{\pi}{4-\pi}} - \hat{C}'_{L_Y}\right)\frac{\Gamma(m-\frac{1}{2})}{\sqrt{2}L_Y\sqrt{\frac{2}{4-\pi}}\Gamma(m)}$$

Then,

$$\Rightarrow P\left(C_{L_Y} \ge \sqrt{\frac{\pi}{4-\pi}} - \left(\frac{\pi}{4-\pi} - \hat{C}'_{L_Y}\right)\frac{\Gamma(m-\frac{1}{2})}{\Gamma(m)}\left(\frac{CHIINV(1-\alpha, 2m)}{2}\right)^{\frac{1}{2}}\right) = 1-\alpha \tag{5.3}$$

From (5.3), then

$$C_{L_Y} \geqslant \sqrt{\frac{\pi}{4-\pi}} - \left(\frac{\pi}{4-\pi} - \hat{C}'_{L_Y}\right)\frac{\Gamma(m-\frac{1}{2})}{\Gamma(m)}\left(\frac{CHIINV(1-\alpha, 2m)}{2}\right)^{\frac{1}{2}}$$

is the level $(1-\alpha)$ one-sided confidence interval for C_{L_Y}.
Thus, the level $(1-\alpha)$ lower confidence bound for C_L can be written as:

$$LB = \sqrt{\frac{\pi}{4-\pi}} - \left(\frac{\pi}{4-\pi} - \hat{C}'_{L_Y}\right)\frac{\Gamma(m-\frac{1}{2})}{\Gamma(m)}\left(\frac{CHIINV(1-\alpha, 2m)}{2}\right)^{\frac{1}{2}} \tag{5.4}$$

where the $\hat{C}'_{L_Y}$, α and m denote the MLE of C_{L_Y}, the specified significance level and the number of observed failures before termination respectively.

6. The Monte Carlo Simulation Study

First, we will show the results of a simulation study for confidence level $(1-\alpha)$ based on a $100(1-\alpha)\%$ one-sided confidence interval of the lifetime performance index C_{L_Y}. Samples from a two parameter Rayleigh distribution were generated under progressive type II right censored sample at $\alpha = 0.01$.

The Monte Carlo simulation algorithm of confidence level $(1-\alpha)$ is given in the following steps:

Step 1: Given n, m, a, b, L, α and $r = (r_1, r_2, ..., r_m)$, where $\alpha > 0, b > 0$ and $m \le n$.

Step 2: (a) Generate data $U_1, U_2, ..., U_m$ by uniform distribution U(0,1).

(b) By the transformation of $Z_i = -ln(1-U_i)$, $i = 1, 2, ..., m$, $(Z_1, Z_2, ...Z_m)$ is a random sample from the exponential distribution.

(c) Set

$$X_{i:m:n} = \frac{Z_1}{n} + \frac{Z_2}{n - R_1 - 1} + ... + \frac{Z_i}{n - R_1 - R_2 ... R_{i-1} - i - 1}, \text{ for } i = 1, 2, .., m.$$

$(X_{1:m:n} X_{2:m:n}, ..., X_{i:m:n})$ is the progressively type II right censored sample from a one parameter exponential distribution.

(d) Finally, set

$$X_{i,n} = F^{-1}[1 - exp(-X'_{i,n})], \text{ for } i = 1, 2, ..., m$$

where $F^{-1}()$ is the inverse cumulative distribution function of the two-parameter Rayleigh distribution. Then, $X_{1:m:n}, X_{2:m:n}, ..., X_{i:m:n}$ is the required progressively type II right censored sample from two parameter Rayleigh distribution.

Step 3: Now apply the transformation $Y_{i:m:n} = \sqrt{2}\lambda^{\frac{3}{2}}(X_{i:m:n} - \mu)$ where $Y_{i:m:n}$ is the progressively type II right censored sample from a one parameter Rayleigh distribution.

Step 4: (a) Calculate the level $100(1-\alpha)\%$ one-sided confidence interval $[LB, \infty)$ for C_{L_Y}, where

$$\underline{LB} = \sqrt{\frac{\pi}{4-\pi} - \left(\frac{\pi}{4-\pi} - \hat{C}'_{L_Y}\right) \frac{\Gamma(m - \frac{1}{2})}{\Gamma(m)} \left(\frac{CHIINV(1-\alpha, 2m)}{2}\right)^{\frac{1}{2}}}$$

where $\hat{C'}_{L_Y}$ is given by (4.6).

(b) If $C_L \geqslant \underline{LB}$ then Count A = 1, else Count A = 0.

Step 5: (a) Steps 2-4 are repeted 100 times.

(b) The estimation of confidence level $(1-\alpha)$ is $(1-\hat{\alpha}) = \frac{\text{Total Count A}}{100}$ for one-sided confidence interval.

Step 6: (a) Repeat steps 2-5 1000 times, then the 1000 estimations of confidence levels are as follows: $(1-\hat{\alpha})_1, (1-\hat{\alpha})_2, ..., (1-\hat{\alpha})_{1000}$ for one-sided confidence interval.

(b) The average empirical confidence interval is $\overline{1-\alpha} = \frac{1}{1000}\sum_{i=1}^{1000}(1-\hat{\alpha})_i$ for one-sided confidence interval.

(c) The sample mean square error (SMSE) of $(1-\hat{\alpha})_1, (1-\hat{\alpha})_2, ..., (1-\hat{\alpha})_{1000}$, $SMSE = (1/1000)\sum_{i=1}^{1000}[(1-\hat{\alpha})_i - (1-\alpha)]^2$ for one-sided confidence inverval.

R statistical software is utilized to calculate the average empirical confidence level and the sample mean square error (SMSE) based on the above Monte Carlo simulation algorithm. Moreover, the average empirical confidence level $\overline{1-\alpha}$ is used as the Monte Carlo estimate of the confidence level $1-\alpha$.

Simulation results are summarized in Table 6.1 for the different n and m $(n \geqslant m)$, censoring scheme $r = (r_1, r_2, ..., r_m)$ where the prior parameter (a,b), L=1.0 and $\alpha = 0.01$.

TABLE 6.1. Average empirical confidence level $(1-\alpha)$ for C_{L_Y} at $\alpha = 0.01(\alpha = 0.05)$.

n	m	$r = (r1, r2, ..., r_m)$	SMSE	AVECL
		(0,0,0,0,10)		
10	5	(10,0,0,0,0)	9.48e-05	0.98988
		(0,5,5,0,0)	(0.0004490)	(0.95106)
		(2,2,2,2,2)		
	5	(0,4,2,3,1)	9.48e-05	0.98988
		(1,1,3,3,2)	(0.0004490)	(0.95106)
15		(9*0,5)		
	10	(5,9*0)	9.67e-05	0.98993
		(0*8,3,2)	(0.0004736)	(0.94958)
		(3,3,3,3,3)		
	5	(3,0,2,1,9)	9.48e-05	0.98988
		(5,4,3,2,1)	(0.0004490)	(0.95106)
		(3,0,7,7*0)		
20	10	(9*0,10)	9.67e-05	0.98993
		(10,9*0)	(0.0004736)	(0.94958)
		(14*0,5)		
	15	(5,14*0)	9.3e-05	0.99042
		(0,2,3,12*0)	(0.0004602)	(0.95030)
		(20,0,0,0,0)		
	5	(0,10,5,5,0)	9.48e-05	0.98988
		(5,5,5,5,0)	(0.0004490)	(0.95106)
		(15,9*0)		
	10	(7*0,5,5,5)	9.67e-05	0.98993
25		(9*0,15)	(0.0004736)	(0.94958)
		(10,14*0)		
	15	(13*0,5,5)	9.3e-05	0.99042
		(14*0,10)	(0.0004602)	(0.95030)
		(19*0,5)		
	20	(18*0,2,3)	9.56e-05	0.99038
		(5,19*0)	(0.0004517)	(0.95003)
		(35,0,0,0,0)		
	5	(0,0,0,0,35)	9.48e-05	0.98988
		(10,10,10,5,0)	(0.0004490)	(0.95106)
		(7*0,10,10,10)		
	10	(9*0,30)	9.67e-05	0.98993
		(30,9*0)	(0.0004736)	(0.94958)
		(14*0,25)		
	15	(25,14*0)	9.3e-05	0.99042
		(13*0,15,10)	(0.0004602)	(0.95030)
		(20,19*0)		
	20	(19*0,20)	9.56e-05	0.99038
		(18*0,10,10)	(0.0004517)	(0.95003)
40		(24*0,15)		
	25	(15,24*0)	9.13e-05	0.99015
		(22*0,5,5,5)	(0.0004679)	(0.94991)
		(10,29*0)		
	30	(28*0,5,5)	9.01e-05	0.99053
		(29*0,10)	(0.0004841)	(0.95047)
		(34*0,5)		
	35	(5,34*0)	8.54e-05	0.99042
		(33*0,2,3)	(0.0004280)	(0.95096)
		(9*0,40)		
	10	(40,9*0)	9.67e-05	0.98993
		(7*0,10,10,20)	(0.0004736)	(0.94958)
		(30,19*0)		
	20	(19*0,30)	9.56e-05	0.99038
		(17*0,10,10,10)	(0.0004517)	(0.95003)
		(25,24*0)		
50	25	(24*0,25)	9.13e-05	0.99015
		(22*0,10,10,5)	(0.0004679)	(0.94991)
		(29*0,20)		
	30	(20,29*0)	9.01e-05	0.99053
		(10,10,28*0)	(0.0004841)	(0.95047)
		(10,39*0)		
	40	(39*0,10)	8.96e-05	0.99064
		(5,5,38*0)	(0.0004444)	(0.94976)

Based on the above results, we can see that all of the average empirical confidence level $\overline{1-\alpha}$ are very close to confidence level $(1-\alpha)$ for any observed number m. For any fixed observed number m, then the average empirical confidence level $\overline{1-\alpha}$ and the corresponding value of SMSE will be the same, respectively, for any n. All of the average empirical confidence levels have a small SMSE.

7. Conclusion

Process capability indices are widely used by manufacturers to measure the potential performance of their processes. In general, in life testing experiments, the experimenter may not always be in a position to observe the lifetimes of all test products because of time limitation and/or restrictions on data collection. Therefore, use of censored samples are recommended. In this article, we consider the lifetime performance index C_L of products with two-parameter Rayleigh distribution under a progressively type 11 right censored sample. Two-parameter Rayleigh distribution was transformed to one-parameter Rayleigh distribution. The MLE of the parameter λ of the one parameter Rayleigh distribution was estimated. MLE of C_L can be calculated by substituting the MLE of λ to (2.6). Moreover, the MLE of C_L is utilized to develop a new hypothesis procedure for the lifetime performance index. In this hypothesis procedure, we first estimated the critical value of the test. This critical value was used to estimate the lower confidence bound for C_L. In the simulation study, the results for the average empirical confidence level $\overline{1-\alpha}$ are very close to confidence level $(1-\alpha)$ for any observed number n. For fixed observed number m, the empirical confidence level $\overline{1-\alpha}$ and the corresponding value of SMSE will be the same, respectively, for any n. This suggest that the proposed testing procedure not only can be easily applied but also can effectively evaluate whether the lifetime of product adheres to the required level.

Acknowledgement

We would like to recognize Mr. Nikesh Subedi for his contributions to programming and Mr. Mike Ingram for constructive criticism of the manuscript. The authors are also very much grateful to the editor and reviewers for their suggestions and helpful comments which led to the improvement of this paper.

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TABLE .1. Critical value C_0 for m=1(1)40 and c=0.1(0.1)0.9 at $\alpha = 0.01$

k	c=.1	c=.2	c=.3	c=.4	c=.5	c=.6	c=.7	c=.8	c=.9
1	1.3221	1.3547	1.3873	1.4199	1.4524	1.4850	1.5176	1.5502	1.5828
2	0.9738	1.0256	1.0774	1.1292	1.1810	1.2328	1.2846	1.3364	1.3882
3	0.8259	0.8859	0.9459	1.0058	1.0658	1.1257	1.1857	1.2457	1.3056
4	0.7375	0.8024	0.8672	0.932	0.9969	1.0617	1.1265	1.1914	1.2562
5	0.6766	0.7448	0.8130	0.8812	0.9494	1.0176	1.0858	1.1540	1.2222
6	0.6311	0.7018	0.7725	0.8432	0.9139	0.9846	1.0553	1.1261	1.1968
7	0.5954	0.6681	0.7407	0.8134	0.8861	0.9588	1.0315	1.1041	1.1768
8	0.5663	0.6406	0.7149	0.7892	0.8635	0.9377	1.0120	1.0863	1.1606
9	0.5421	0.6177	0.6933	0.7689	0.8445	0.9202	0.9958	1.0714	1.1470
10	0.5214	0.5981	0.6749	0.7517	0.8284	0.9052	0.9819	1.0587	1.1354
11	0.5035	0.5812	0.6589	0.7367	0.8144	0.8922	0.9699	1.0477	1.1254
12	0.4877	0.5663	0.6450	0.7236	0.8022	0.8808	0.9594	1.0380	1.1166
13	0.4738	0.5532	0.6325	0.7119	0.7913	0.8707	0.9501	1.0295	1.1088
14	0.4613	0.5413	0.6214	0.7015	0.7816	0.8616	0.9417	1.0218	1.1019
15	0.4500	0.5307	0.6114	0.6921	0.7728	0.8535	0.9342	1.0149	1.0956
16	0.4397	0.5210	0.6023	0.6835	0.7648	0.8460	0.9273	1.0086	1.0898
17	0.4304	0.5121	0.5939	0.6757	0.7575	0.8393	0.9210	1.0028	1.0846
18	0.4218	0.5040	0.5863	0.6685	0.7508	0.8330	0.9153	0.9975	1.0798
19	0.4138	0.4965	0.5792	0.6619	0.7446	0.8273	0.9100	0.9926	1.0753
20	0.4064	0.4895	0.5726	0.6557	0.7388	0.8219	0.905	0.9881	1.0712
21	0.3996	0.4831	0.5665	0.6500	0.7335	0.8170	0.9004	0.9839	1.0674
22	0.3932	0.4770	0.5608	0.6447	0.7285	0.8123	0.8961	0.9800	1.0638
23	0.3872	0.4713	0.5555	0.6397	0.7238	0.8080	0.8921	0.9763	1.0605
24	0.3815	0.4660	0.5505	0.6349	0.7194	0.8039	0.8884	0.9728	1.0573
25	0.3762	0.4610	0.5458	0.6305	0.7153	0.8000	0.8848	0.9696	1.0543
26	0.3712	0.4563	0.5413	0.6263	0.7114	0.7964	0.8815	0.9665	1.0515
27	0.3665	0.4518	0.5371	0.6224	0.7077	0.7930	0.8783	0.9636	1.0489
28	0.3620	0.4475	0.5331	0.6186	0.7042	0.7897	0.8753	0.9608	1.0464
29	0.3577	0.4435	0.5293	0.6151	0.7009	0.7866	0.8724	0.9582	1.0440
30	0.3537	0.4397	0.5257	0.6117	0.6977	0.7837	0.8697	0.9557	1.0417
31	0.3498	0.4360	0.5222	0.6085	0.6947	0.7809	0.8671	0.9533	1.0396
32	0.3461	0.4325	0.5189	0.6054	0.6918	0.7782	0.8647	0.9511	1.0375
33	0.3426	0.4292	0.5158	0.6024	0.6890	0.7757	0.8623	0.9489	1.0355
34	0.3392	0.4260	0.5128	0.5996	0.6864	0.7732	0.8600	0.9468	1.0336
35	0.3359	0.4229	0.5099	0.5969	0.6839	0.7709	0.8579	0.9448	1.0318
36	0.3328	0.4200	0.5072	0.5943	0.6815	0.7686	0.8558	0.9429	1.0301
37	0.3299	0.4172	0.5045	0.5918	0.6791	0.7665	0.8538	0.9411	1.0284
38	0.3270	0.4145	0.5019	0.5894	0.6769	0.7644	0.8519	0.9393	1.0268
39	0.3242	0.4119	0.4995	0.5871	0.6747	0.7624	0.8500	0.9376	1.0253
40	0.3216	0.4093	0.4971	0.5849	0.6727	0.7605	0.8482	0.9360	1.0238
41	0.3190	0.4069	0.4948	0.5828	0.6707	0.7586	0.8465	0.9344	1.0224
42	0.3165	0.4046	0.4926	0.5807	0.6687	0.7568	0.8449	0.9329	1.0210
43	0.3141	0.4023	0.4905	0.5787	0.6669	0.7551	0.8433	0.9314	1.0196
44	0.3118	0.4001	0.4884	0.5768	0.6651	0.7534	0.8417	0.9300	1.0183
45	0.3096	0.3980	0.4864	0.5749	0.6633	0.7518	0.8402	0.9287	1.0171
46	0.3074	0.3960	0.4845	0.5731	0.6616	0.7502	0.8388	0.9273	1.0159
47	0.3053	0.3940	0.4826	0.5713	0.6600	0.7487	0.8374	0.9260	1.0147
48	0.3033	0.3920	0.4808	0.5696	0.6584	0.7472	0.8360	0.9248	1.0136
49	0.3013	0.3902	0.4791	0.5680	0.6569	0.7458	0.8347	0.9236	1.0125
50	0.2994	0.3884	0.4774	0.5664	0.6554	0.7444	0.8334	0.9224	1.0114

TABLE .2. Critical value C_0 for m=1(1)40 and c=0.1(0.1)0.9 at $\alpha = 0.05$

k	c=.1	c=.2	c=.3	c=.4	c=.5	c=.6	c=.7	c=.8	c=.9
1	1.4364	1.4627	1.489	1.5153	1.5416	1.5678	1.5941	1.6204	1.6467
2	1.1190	1.1628	1.2066	1.2504	1.2942	1.338	1.3818	1.4256	1.4694
3	0.9722	1.0241	1.076	1.1279	1.1798	1.2317	1.2836	1.3355	1.3874
4	0.8803	0.9372	0.9942	1.0512	1.1081	1.1651	1.2221	1.2790	1.336
5	0.8149	0.8755	0.936	0.9966	1.0572	1.1178	1.1783	1.2389	1.2995
6	0.765	0.8283	0.8917	0.955	1.0183	1.0816	1.1449	1.2083	1.2716
7	0.7251	0.7907	0.8562	0.9217	0.9872	1.0527	1.1183	1.1838	1.2493
8	0.6922	0.7596	0.8269	0.8942	0.9616	1.0289	1.0963	1.1636	1.2309
9	0.6644	0.7333	0.8022	0.8710	0.9399	1.0088	1.0776	1.1465	1.2154
10	0.6405	0.7107	0.7809	0.8511	0.9213	0.9914	1.0616	1.1318	1.2020
11	0.6196	0.6909	0.7623	0.8336	0.9050	0.9763	1.0476	1.1190	1.1903
12	0.6011	0.6734	0.7458	0.8182	0.8905	0.9629	1.0353	1.1076	1.1800
13	0.5846	0.6578	0.7311	0.8044	0.8777	0.9509	1.0242	1.0975	1.1708
14	0.5697	0.6438	0.7179	0.7920	0.8661	0.9402	1.0142	1.0883	1.1624
15	0.5562	0.6310	0.7059	0.7807	0.8555	0.9304	1.0052	1.0801	1.1549
16	0.5438	0.6194	0.6949	0.7704	0.8459	0.9214	0.9970	1.0725	1.1480
17	0.5325	0.6087	0.6848	0.7610	0.8371	0.9132	0.9894	1.0655	1.1417
18	0.5221	0.5988	0.6755	0.7522	0.8290	0.9057	0.9824	1.0591	1.1358
19	0.5124	0.5896	0.6669	0.7441	0.8214	0.8987	0.9759	1.0532	1.1304
20	0.5034	0.5811	0.6589	0.7366	0.8144	0.8921	0.9699	1.0476	1.1254
21	0.4950	0.5732	0.6514	0.7296	0.8078	0.8860	0.9643	1.0425	1.1207
22	0.4871	0.5657	0.6444	0.7230	0.8017	0.8803	0.9590	1.0376	1.1163
23	0.4797	0.5587	0.6378	0.7169	0.7959	0.8750	0.9540	1.0331	1.1121
24	0.4727	0.5522	0.6316	0.7110	0.7905	0.8699	0.9494	1.0288	1.1083
25	0.4661	0.5459	0.6257	0.7055	0.7854	0.8652	0.9450	1.0248	1.1046
26	0.4599	0.5400	0.6202	0.7003	0.7805	0.8606	0.9408	1.0209	1.1011
27	0.4540	0.5345	0.6149	0.6954	0.7759	0.8564	0.9368	1.0173	1.0978
28	0.4484	0.5292	0.6100	0.6907	0.7715	0.8523	0.9331	1.0139	1.0947
29	0.4430	0.5241	0.6052	0.6863	0.7674	0.8484	0.9295	1.0106	1.0917
30	0.4380	0.5193	0.6007	0.6820	0.7634	0.8448	0.9261	1.0075	1.0888
31	0.4331	0.5147	0.5964	0.6780	0.7596	0.8412	0.9229	1.0045	1.0861
32	0.4285	0.5103	0.5922	0.6741	0.7560	0.8379	0.9198	1.0016	1.0835
33	0.4240	0.5061	0.5883	0.6704	0.7525	0.8347	0.9168	0.9989	1.0810
34	0.4197	0.5021	0.5845	0.6668	0.7492	0.8316	0.9139	0.9963	1.0787
35	0.4156	0.4982	0.5808	0.6634	0.7460	0.8286	0.9112	0.9938	1.0764
36	0.4117	0.4945	0.5773	0.6601	0.7429	0.8257	0.9086	0.9914	1.0742
37	0.4079	0.4909	0.5740	0.6570	0.7400	0.8230	0.9060	0.9890	1.0721
38	0.4043	0.4875	0.5707	0.6539	0.7372	0.8204	0.9036	0.9868	1.0700
39	0.4008	0.4842	0.5676	0.6510	0.7344	0.8178	0.9012	0.9846	1.0681
40	0.3974	0.4810	0.5646	0.6482	0.7318	0.8154	0.8990	0.9826	1.0662
41	0.3941	0.4779	0.5617	0.6454	0.7292	0.813	0.8968	0.9806	1.0643
42	0.3910	0.4749	0.5589	0.6428	0.7268	0.8107	0.8947	0.9786	1.0626
43	0.3879	0.4720	0.5561	0.6403	0.7244	0.8085	0.8926	0.9767	1.0609
44	0.3849	0.4692	0.5535	0.6378	0.7221	0.8064	0.8906	0.9749	1.0592
45	0.3821	0.4665	0.5510	0.6354	0.7198	0.8043	0.8887	0.9732	1.0576
46	0.3793	0.4639	0.5485	0.6331	0.7177	0.8023	0.8869	0.9715	1.0561
47	0.3766	0.4613	0.5461	0.6308	0.7156	0.8003	0.8851	0.9698	1.0545
48	0.3740	0.4589	0.5438	0.6286	0.7135	0.7984	0.8833	0.9682	1.0531
49	0.3714	0.4565	0.5415	0.6265	0.7116	0.7966	0.8816	0.9666	1.0517
50	0.3690	0.4541	0.5393	0.6245	0.7096	0.7948	0.8800	0.9651	1.0503