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Sectorial Stieltjes functions and their realizations by L-systems with a Schrödinger operator

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Dedicated to Eduard Tsekanovskiĭ, my mentor, co-author, and friend, on the occasion of his 75th birthday

We consider classes of sectorial Stieltjes functions. It is shown that a function belonging to these classes can be realized as the impedance function of a singular L-system with a sectorial state-space operator. We provide an additional condition on a given function from this class so that the state-space operator of the realizing L-system is α-sectorial with the exact angle of sectoriality α. Then these results are applied to L-systems based upon a non-self-adjoint Schrödinger operator.

1 Introduction

An operator-valued function $V(z)$ acting on a finite-dimensional Hilbert space E belongs to the class of operator-valued Herglotz-Nevanlinna functions if it is holomorphic on $\mathbb{C}\setminus\mathbb{R}$, if it is symmetric with respect to the real axis, i.e., $V(z)^* = V(\bar z)$, $z\in\mathbb{C}\setminus\mathbb{R}$, and if it satisfies the positivity condition

$$\operatorname{Im} V(z) \geq 0, \quad z \in \mathbb{C}_+.$$

It is well-known (see e.g. [7], [8]) that operator-valued Herglotz-Nevanlinna functions admit the following integral representation:

$$V(z) = Q + Lz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) dG(t), \quad z \in \mathbb{C}\setminus\mathbb{R}, \tag{1.1}$$

where $Q = Q^*$, $L \geq 0$, and $G(t)$ is a nondecreasing operator-valued function on $\mathbb{R}$ with values in the class of nonnegative operators in E such that

$$\int_{\mathbb{R}} \frac{(dG(t)f,f)_E}{1+t^2} < \infty, \quad \forall f \in E.$$

The realization of a selected class of Herglotz-Nevanlinna functions is provided by an L-system Θ of the form

$$\begin{cases} (\mathbb{A} - zI)x = KJ\varphi_-, \\ \varphi_+ = \varphi_- - 2iK^*x \end{cases} \tag{1.2}$$

or

$$\Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & E \end{pmatrix}. \tag{1.3}$$

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In this system $\mathbb{A}$, the *state-space operator* of the system, is a so-called $(*)$-extension, which is a bounded linear operator from $\mathcal{H}_+$ into $\mathcal{H}_-$ extending a symmetric operator A in $\mathcal{H}$, where $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is a rigged Hilbert space. Moreover, K is a bounded linear operator from the finite-dimensional Hilbert space E into $\mathcal{H}_-$, while $J = J^* = J^{-1}$ is acting on E, are such that $\operatorname{Im}\mathbb{A} = KJK^*$. Also, $\varphi_- \in E$ is an input vector, $\varphi_+ \in E$ is an output vector, and $x \in \mathcal{H}_+$ is a vector of the state space of the system Θ. The system described by (1.2)–(1.3) is called an *L-system*. An unbounded generalization of Brodskiĭ–Livšic operator colligations [10], [16], the L-systems have been introduced by Eduard Tsekanovskiĭ, and studied by himself, his students, and co-authors for the last four decades. The detailed description of L-systems including historical aspects can be found in [4]. An operator-valued function

$$W_\Theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ$$

is the transfer function of the L-system Θ. It was shown in [7] that an operator-valued function $V(z)$ acting on a Hilbert space E of the form (1.1) can be represented and realized in the form

$$V(z) = i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I] = K^*(\operatorname{Re}\mathbb{A} - zI)^{-1}K,$$

where $W_\Theta(z)$ is a transfer function of some scattering $(J = I)$ L-system Θ, if and only if the function $V(z)$ in (1.1) satisfies the following two conditions:

$$\begin{cases} L = 0, \\ Qf = \displaystyle\int_{\mathbb{R}} \frac{t}{1+t^2}\, dG(t)f, \quad \text{when} \quad \displaystyle\int_{\mathbb{R}} (dG(t)f, f)_E < \infty. \end{cases} \tag{1.4}$$

The class of all realizable Herglotz-Nevanlinna functions with conditions (1.4) is denoted by $N(R)$ (see [7]).

In the current paper we are going to focus on an important subclass of Herglotz-Nevanlinna functions, the Stieltjes functions. A Herglotz-Nevanlinna function $V(z)$ belongs to the Stieltjes functions subclass if it is holomorphic in $\operatorname{Ext}[0, +\infty)$ and is such that $\operatorname{Im}[zV(z)]/\operatorname{Im} z \geq 0$, i.e., $zV(z)$ is also a Herglotz-Nevanlinna function. The formal definition, integral representation for Stieltjes functions as well as the basic realization results are given in Section 3. In particular, we specify a subclass of realizable Stieltjes operator-functions and show that any member of this subclass can be realized by an L-system of the form (1.3) whose state-space operator $\mathbb{A}$ is accretive.

In Section 4 we introduce the so-called *sectorial* classes S^α and S^{α_1,α_2} of Stieltjes functions. The class S^α was first introduced and treated by Alpay and Tsekanovskiĭ in [2] while the description of the class S^{α_1,α_2} can only be found in [4]. The realization results presented in Section 4 for these sectorial classes allow us to observe the properties of the realizing L-systems whose impedance functions belong to either S^α or S^{α_1,α_2}.

Section 5 is devoted to L-systems of the form (1.3) containing the Schrödinger operator in $L_2[a, +\infty)$ (see [18]) with non-self-adjoint boundary conditions

$$\begin{cases} T_h y = -y'' + q(x)y, \\ y'(a) = hy(a), \end{cases} \quad \big(q(x) = \overline{q(x)},\ \operatorname{Im} h \neq 0\big). \tag{1.5}$$

A complete description of such L-systems as well as the formulas for their transfer and impedance functions are presented. Moreover, Theorem 5.1 provides us with the formula giving the exact parametrization of all state-space operators of L-systems based upon the Schrödinger operator (1.5).

Section 6 contains the main results of the present paper. Utilizing the general realization theorems for the class S^{α_1,α_2} covered in Section 4, we obtain some interesting properties of L-systems with Schrödinger operator whose impedance function fall into the class S^{α_1,α_2}. Most of the results are given in terms of the real parameter μ that appears in the construction of the elements of the realizing system.

2 Preliminaries

For a pair of Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ we denote by $[\mathcal{H}_1, \mathcal{H}_2]$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\dot{A}$ be a closed, densely defined, symmetric operator in a Hilbert space $\mathcal{H}$ with inner product (f, g), $f, g \in \mathcal{H}$.

Any operator T in $\mathcal{H}$ such that

$$\dot{A} \subset T \subset \dot{A}^*$$

is called a *quasi-self-adjoint extension* of $\dot{A}$.

Consider the rigged Hilbert space (see [7]) $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$, where $\mathcal{H}_+ = \mathrm{Dom}\big(\dot{A}^*\big)$ and

$$(f,g)_+ = (f,g) + \big(\dot{A}^* f, \dot{A}^* g\big), \quad f,g \in \mathrm{Dom}(A^*).$$

Let $\mathcal{R}$ be the *Riesz-Berezansky operator* $\mathcal{R}$ (see [7]) which maps $\mathcal{H}_-$ onto $\mathcal{H}_+$ such that $(f,g) = (f,\mathcal{R}g)_+$ $(\forall f \in \mathcal{H}_+,\ g \in \mathcal{H}_-)$ and $\|\mathcal{R}g\|_+ = \|g\|_-$. Note that identifying the space conjugate to $\mathcal{H}_\pm$ with $\mathcal{H}_\mp$, we get that if $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$, then $\mathbb{A}^* \in [\mathcal{H}_+, \mathcal{H}_-]$.

Definition 2.1 An operator $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a *self-adjoint bi-extension* of a symmetric operator $\dot{A}$ if $\mathbb{A} = \mathbb{A}^*$ and $\mathbb{A} \supset \dot{A}$.

Let $\mathbb{A}$ be a self-adjoint bi-extension of $\dot{A}$ and let the operator $\widehat{A}$ in $\mathcal{H}$ be defined as follows:

$$\mathrm{Dom}\big(\widehat{A}\big) = \big\{f \in \mathcal{H}_+ : \widehat{A} f \in \mathcal{H}\big\}, \quad \widehat{A} = \mathbb{A}\restriction \mathrm{Dom}\big(\widehat{A}\big).$$

The operator $\widehat{A}$ is called a *quasi-kernel* of a self-adjoint bi-extension $\mathbb{A}$ (see [21]).

Definition 2.2 Let T be a quasi-self-adjoint extension of $\dot{A}$ with nonempty resolvent set $\rho(T)$. An operator $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a $(*)$*-extension* of an operator T if

(1) $\mathbb{A} \supset T \supset \dot{A}, \quad \mathbb{A}^* \supset T^* \supset \dot{A}$,

(2) the quasi-kernel of self-adjoint bi-extension $\mathrm{Re}\,\mathbb{A} = \frac{1}{2}(\mathbb{A} + \mathbb{A}^*)$ is a self-adjoint extension of $\dot{A}$.

A definition of $(*)$-extension in an equivalent form was first introduced by Eduard Tsekanovskiĭ in [17]. The existence, description, and analog of von Neumann's formulas for self-adjoint bi-extensions and $(*)$-extensions were discussed in [21] (see also [3]–[5], [7]). In what follows we suppose that $\dot{A}$ has equal deficiency indices and will say that a quasi-self-adjoint extension T of $\dot{A}$ belongs to the *class* $\Lambda\big(\dot{A}\big)$ if $\rho(T) \neq \emptyset$, $\mathrm{Dom}\big(\dot{A}\big) = \mathrm{Dom}(T) \cap \mathrm{Dom}(T^*)$, and T admits $(*)$-extensions.

Recall that a linear operator T in a Hilbert space $\mathfrak{H}$ is called *accretive* [15] if $\mathrm{Re}\,(Tf,f) \geq 0$ for all $f \in \mathrm{Dom}(T)$. We call an accretive operator T α*-sectorial* [15] if there exists a value of $\alpha \in (0, \pi/2)$ such that

$$|\,\mathrm{Im}(Tf,f)| \leq (\tan\alpha)\,\mathrm{Re}\,(Tf,f), \quad f \in \mathrm{Dom}(T).$$

We say that the angle of sectoriality α is *exact* for an α-sectorial operator T if

$$\tan\alpha = \sup_{f \in \mathrm{Dom}(T)} \frac{|\,\mathrm{Im}(Tf,f)|}{\mathrm{Re}\,(Tf,f)}.$$

Let T be a quasi-self-adjoint maximal accretive extension of a nonnegative operator $\dot{A}$. A $(*)$-extension $\mathbb{A}$ of T is called *accretive* if $\mathrm{Re}\,(\mathbb{A}f,f) \geq 0$ for all $f \in \mathcal{H}_+$. This is equivalent to that the real part $\mathrm{Re}\,\mathbb{A} = (\mathbb{A} + \mathbb{A}^*)/2$ is nonnegative self-adjoint bi-extension of $\dot{A}$.

Definition 2.3 A system of equations

$$\begin{cases} (\mathbb{A} - zI)x = KJ\varphi_-, \\ \varphi_+ = \varphi_- - 2iK^* x, \end{cases}$$

or an array

$$\Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & E \end{pmatrix} \tag{2.1}$$

is called an *L-system* if:

(1) $\mathbb{A}$ is a $(*)$-extension of an operator T of the class $\Lambda(\dot{A})$;

(2) $J = J^* = J^{-1} \in [E, E], \quad \dim E < \infty$;

(3) $\operatorname{Im}\mathbb{A} = KJK^*$, where $K \in [E, \mathcal{H}_-]$, $K^* \in [\mathcal{H}_+, E]$, and $\operatorname{Ran}(K) = \operatorname{Ran}(\operatorname{Im}\mathbb{A})$.

In the definition above $\varphi_- \in E$ stands for an input vector, $\varphi_+ \in E$ is an output vector, and x is a state space vector in $\mathcal{H}$. An operator $\mathbb{A}$ is called a *state-space operator* of the system Θ, J is a *direction operator*, and K is a *channel operator*. A system Θ of the form (2.1) is called an *accretive system* [9], [12] if its main operator $\mathbb{A}$ is accretive.

We associate with an L-system Θ the operator-valued function

$$W_\Theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ, \quad z \in \rho(T), \tag{2.2}$$

which is called the *transfer function* of the L-system Θ. We also consider the operator-valued function

$$V_\Theta(z) = K^*(\operatorname{Re}\mathbb{A} - zI)^{-1}K. \tag{2.3}$$

It was shown in [7], [4] that both (2.2) and (2.3) are well defined. The transfer operator-function $W_\Theta(z)$ of the system Θ and an operator-function $V_\Theta(z)$ of the form (2.3) are connected by the following relations valid for $\operatorname{Im} z \neq 0$, $z \in \rho(T)$,

$$\begin{aligned} V_\Theta(z) &= i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I]J, \\ W_\Theta(z) &= (I + iV_\Theta(z)J)^{-1}(I - iV_\Theta(z)J). \end{aligned}$$

The function $V_\Theta(z)$ defined by (2.3) is called the *impedance function* of an L-system Θ of the form (2.1). It was shown in [7] that the class $N(R)$ of all Herglotz-Nevanlinna functions in a finite-dimensional Hilbert space E, that can be realized as impedance functions of an L-system, is described by conditions (1.4). In particular, the following theorem [4], [7] takes place.

Theorem 2.4 *Let Θ be an L-system of the form* (2.1). *Then the impedance function $V_\Theta(z)$ of the form* (2.3) *belongs to the class $N(R)$.*

Conversely, let an operator-valued function $V(z)$ belong to the class $N(R)$. Then $V(z)$ can be realized as the impedance function of an L-system Θ of the form (2.1) *with a preassigned direction operator J for which $I + iV(-i)J$ is invertible.*

It was shown in [7] that if $J = I$, then the invertibility condition in the second part of Theorem 2.4 is satisfied automatically.

3 Realization of Stieltjes functions

Let E be a finite-dimensional Hilbert space. The scalar versions of the following definition can be found in [14].

Definition 3.1 We will call an operator-valued Herglotz-Nevanlinna function $V(z) \in [E, E]$ a *Stieltjes function* if $V(z)$ admits the following integral representation

$$V(z) = \gamma + \int_0^\infty \frac{dG(t)}{t - z}, \tag{3.1}$$

where $\gamma \geq 0$ and $G(t)$ is a non-decreasing on $[0, +\infty)$ operator-valued function such that

$$\int_0^\infty \frac{(dG(t)f, f)_E}{1 + t} < \infty, \quad \forall f \in E.$$

Alternatively (see [14]) an operator-valued function $V(z)$ is Stieltjes if it is holomorphic in $\operatorname{Ext}[0, +\infty)$ and

$$\frac{\operatorname{Im}[zV(z)]}{\operatorname{Im} z} \geq 0. \tag{3.2}$$

Theorem 3.2 below was stated in equivalent ways and proved in [4], [11], [12].

Theorem 3.2 *Let Θ be an L-system of the form* (2.1). *Then the impedance operator-valued function $V_\Theta(z)$ defined by* (2.3) *is a Stieltjes function if and only if the main operator $\mathbb{A}$ of the system Θ is accretive.*

At this point we need to note that since Stieltjes functions form a subset of Herglotz-Nevanlinna functions then we can utilize the conditions (1.4) to form a *class* $S(R)$ of all *realizable Stieltjes functions* presented in [4], [12]. Clearly, $S(R)$ is a subclass of $N(R)$ of all realizable Herglotz-Nevanlinna functions described in details in [7] and [8]. To see the specifications of the class $S(R)$ we recall that aside of integral representation (3.1), any Stieltjes function admits a representation (1.1). Applying condition (1.4) we obtain

$$Q = \frac{1}{2}\big[V_\Theta(-i) + V^*_\Theta(-i)\big] = \gamma + \int\limits_0^{+\infty} \frac{t}{1+t^2}\, dG(t). \tag{3.3}$$

Combining the second part of condition (1.4) and (3.3) we conclude that

$$\gamma f = 0, \tag{3.4}$$

for all $f \in E$ such that

$$\int\limits_0^{\infty} (dG(t)f, f)_E < \infty \tag{3.5}$$

holds. Consequently, (3.4)–(3.5) is precisely the condition for $V(z) \in S(R)$.

We are going to focus though on the subclass $S_0(R)$ of $S(R)$ (see [4], [12]), whose definition is the following.

Definition 3.3 An operator-valued Stieltjes function $V(z) \in [E, E]$ is said to be a member of the *class* $S_0(R)$ if in the representation (3.1) we have

$$\int\limits_0^{\infty} (dG(t)f, f)_E = \infty$$

for all non-zero $f \in E$.

We note that a function $V(z)$ can belong to the class $S_0(R)$ and have an arbitrary constant $\gamma \geq 0$ in the representation (3.1).

The following statement [12] is the direct realization theorem for the functions of the class $S_0(R)$.

Theorem 3.4 *Let Θ be an accretive system of the form* (2.1). *Then the impedance operator-function $V_\Theta(z)$ of the form* (2.3) *belongs to the class $S_0(R)$.*

The inverse realization theorem can be stated and proved (see [12]) for the classes $S_0(R)$ as follows.

Theorem 3.5 *Let an operator-valued function $V(z)$ belong to the class $S_0(R)$. Then $V(z)$ admits a realization by an accretive system Θ of the form* (2.1) *with $J = I$.*

4 Sectorial classes S^{α} and S^{α_1,α_2} and their realizations

Let $\alpha \in \left(0, \frac{\pi}{2}\right)$. We introduce *sectorial subclasses* S^α of operator-valued Stieltjes functions as follows. An operator-valued Stieltjes function $V(z)$ belongs to S^α if

$$K_\alpha = \sum_{k,l=1}^{n} \left(\left[\frac{z_k V(z_k) - \bar{z}_l V(\bar{z}_l)}{z_k - \bar{z}_l} - (\cot\alpha)\, V^*(z_l)V(z_k)\right] h_k, h_l\right)_E \geq 0, \tag{4.1}$$

for an arbitrary sequence $\{z_k\}$ $(k = 1, \dots, n)$ of $(\operatorname{Im} z_k > 0)$ complex numbers and a sequence of vectors $\{h_k\}$ in E. For $0 < \alpha_1 < \alpha_2 < \frac{\pi}{2}$, we have

$$S^{\alpha_1} \subset S^{\alpha_2} \subset S,$$

where S denotes the class of all Stieltjes functions (which corresponds to the case $\alpha = \frac{\pi}{2}$), as follows from the inequality

$$K_{\alpha_1} \leq K_{\alpha_2} \leq K_{\frac{\pi}{2}}.$$

The following theorem [2], [4] refines the result of Theorem 3.2 as applied to the class S^α.

Theorem 4.1 *Let Θ be a scattering L-system of the form* (2.1) *with a densely defined non-negative symmetric operator $\dot{A}$. Then the impedance function $V_\Theta(z)$ defined by* (2.3) *belongs to the class S^α if and only if the operator $\mathbb{A}$ of the L-system Θ is α-sectorial.*

Another class that we would like to introduce at this point is a special subclass of scalar Stieltjes functions. Let

$$0 \le \alpha_1 \le \alpha_2 \le \frac{\pi}{2}.$$

We say that a scalar Stieltjes function $V(z)$ belongs to the *class* S^{α_1,α_2} if

$$\tan\alpha_1 = \lim_{x\to-\infty} V(x), \quad \tan\alpha_2 = \lim_{x\to-0} V(x). \tag{4.2}$$

The following theorem [4] provides a connection between the classes S^α and S^{α_1,α_2}.

Theorem 4.2 *Let Θ be a scattering L-system of the form*

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}, \tag{4.3}$$

with a densely defined non-negative symmetric operator $\dot{A}$. Let also $\mathbb{A}$ be an α-sectorial $(*)$-extension of $T \in \Lambda(\dot{A})$. Then the impedance function $V_\Theta(z)$ defined by (2.3) *belongs to the class S^{α_1,α_2}, $\tan\alpha_2 \le \tan\alpha$, and T is $(\alpha_2-\alpha_1)$-sectorial with the exact angle of sectoriality $(\alpha_2-\alpha_1)$.*

The corollary below treats the case when α in Theorem 4.2 is the exact angle of sectoriality of the operator T. Thus both operators T and $\mathbb{A}$ maintain the same exact angle.

Corollary 4.3 *Let Θ of the form* (4.3) *be an L-system as in the statement of Theorem* 4.2 *and let α be the exact angle of sectoriality of the operator T of the system Θ. Then $V_\Theta(z) \in S^{0,\alpha}$.*

Proof. According to Theorem 4.2 the exact angle of sectoriality is given by $\alpha_2-\alpha_1$, where

$$\tan\alpha_1 = \lim_{x\to-\infty} V_\Theta(x), \quad \tan\alpha_2 = \lim_{x\to-0} V_\Theta(x).$$

It was also shown that $\tan\alpha \ge \tan\alpha_2$. On the other hand, since in the statement of the current corollary α be the exact angle of sectoriality of T, then $\alpha = \alpha_2-\alpha_1$ and hence $\tan(\alpha_2-\alpha_1) \ge \tan\alpha_2$. Therefore, $\alpha_1 = 0$. □

Remark 4.4 It follows that under assumptions of Corollary 4.3, the impedance function $V_\Theta(z)$ has the form

$$V_\Theta(z) = \int_0^\infty \frac{dG(t)}{t-z}.$$

For the remainder of this paper we will need to rely on the following theorem whose proof can be found in [4].

Theorem 4.5 *Let Θ be an L-system of the form* (4.3)*, where $\mathbb{A}$ is a $(*)$-extension of $T \in \Lambda(\dot{A})$ and $\dot{A}$ is a closed densely defined non-negative symmetric operator with deficiency numbers* $(1,1)$. *If the impedance function $V_\Theta(z)$ belongs to the class S^{α_1,α_2}, then $\mathbb{A}$ is α-sectorial, where*

$$\tan\alpha = \tan\alpha_2 + 2\sqrt{\tan\alpha_1(\tan\alpha_2-\tan\alpha_1)}.$$

The next statement gives an explicit description of all the functions from the class S^{α_1,α_2} that are realizable as impedance functions of such L-systems that the exact angles of sectoriality of T and $\mathbb{A}$ coincide. Its proof immediately follows from Theorems 4.2 and 4.5.

Theorem 4.6 *Let Θ be an L-system of the form* (4.3) *with a densely defined non-negative symmetric operator $\dot{A}$. Then $\mathbb{A}$ is α-sectorial $(*)$-extension of an α-sectorial operator $T \in \Lambda(\dot{A})$ with the exact angle $\alpha \in (0,\pi/2)$ if and only if*

$$V_\Theta(z) = \int_0^\infty \frac{dG(t)}{t-z} \in S^{0,\alpha}.$$

Moreover, the angle α can be found via the formula

$$\tan\alpha = \int\limits_0^\infty \frac{dG(t)}{t}. \tag{4.4}$$

5 L-systems with a Schrödinger operator

Let $\mathcal{H} = L_2[a,+\infty)$ and $l(y) = -y'' + q(x)y$, where q is a real locally summable function. Suppose that the symmetric operator

$$\begin{cases} \dot{A}y = -y'' + q(x)y, \\ y(a) = y'(a) = 0 \end{cases} \tag{5.1}$$

has deficiency indices (1,1). Let D^* be the set of functions locally absolutely continuous together with their first derivatives such that $l(y) \in L_2[a,+\infty)$. Consider $\mathcal{H}_+ = \mathrm{Dom}\big(\dot{A}^*\big) = D^*$ with the scalar product

$$(y,z)_+ = \int\limits_a^\infty \big(y(x)\overline{z(x)} + l(y)\overline{l(z)}\,\big)dx, \quad y,\ z \in D^*.$$

Let

$$\mathcal{H}_+ \subset L_2[a,+\infty) \subset \mathcal{H}_-$$

be the corresponding triplet of Hilbert spaces. Consider the operators

$$\begin{cases} T_h y = l(y) = -y'' + q(x)y, \\ hy(a) - y'(a) = 0, \end{cases} \quad \begin{cases} T_h^* y = l(y) = -y'' + q(x)y, \\ \overline{h}y(a) - y'(a) = 0. \end{cases} \tag{5.2}$$

The following theorem was proved in [4], [6].

Theorem 5.1 *The set of all $(*)$-extensions of a non-self-adjoint Schrödinger operator T_h of the form* (5.2) *in $L_2[a,+\infty)$ can be represented in the form*

$$\begin{aligned} \mathbb{A}y &= -y'' + q(x)y - \frac{1}{\mu - h}\,[y'(a) - hy(a)]\,[\mu\delta(x-a) + \delta'(x-a)], \\ \mathbb{A}^*y &= -y'' + q(x)y - \frac{1}{\mu - \overline{h}}\,[y'(a) - \overline{h}y(a)]\,[\mu\delta(x-a) + \delta'(x-a)]. \end{aligned} \tag{5.3}$$

Moreover, the formulas (5.3) *establish a one-to-one correspondence between the set of all $(*)$-extensions of a Schrödinger operator T_h of the form* (5.2) *and all real numbers $\mu \in [-\infty,+\infty]$.*

Let $\dot{A}$ be a symmetric operator of the form (5.1) with deficiency indices (1,1), generated by the differential operation $l(y) = -y'' + q(x)y$. Let also $\varphi_k(x,\lambda)$ $(k = 1,2)$ be the solutions of the following Cauchy problems:

$$\begin{cases} l(\varphi_1) = \lambda\varphi_1, \\ \varphi_1(a,\lambda) = 0, \\ \varphi_1'(a,\lambda) = 1, \end{cases} \quad \begin{cases} l(\varphi_2) = \lambda\varphi_2, \\ \varphi_2(a,\lambda) = -1, \\ \varphi_2'(a,\lambda) = 0, \end{cases}$$

It is well-known [1] that there exists a function $m_\infty(\lambda)$ (called the Weyl-Titchmarsh function) for which

$$\varphi(x,\lambda) = \varphi_2(x,\lambda) + m_\infty(\lambda)\varphi_1(x,\lambda)$$

belongs to $L_2[a,+\infty)$.

Suppose that the symmetric operator $\dot{A}$ of the form (5.1) with deficiency indices (1,1) is nonnegative, i.e., $\big(\dot{A}f, f\big) \geq 0$ for all $f \in \operatorname{Dom}\big(\dot{A}\big)$. It was shown in [19], [20] that the Schrödinger operator T_h of the form (5.2) is accretive if and only if

$$\operatorname{Re} h \geq -m_\infty(-0). \tag{5.4}$$

The following theorem will be needed in the next section. Its proof can be located in [4].

Theorem 5.2 *Let T_h $(\operatorname{Im} h > 0)$ be an accretive Schrödinger operator of the form* (5.2). *Then for all real μ satisfying the following inequality*

$$\mu \geq \frac{(\operatorname{Im} h)^2}{m_\infty(-0) + \operatorname{Re} h} + \operatorname{Re} h,$$

the operators $\mathbb{A}$ in (5.3) *define the set of all accretive $(*)$-extensions $\mathbb{A}$ of the operator T_h. The operator T_h has a unique accretive $(*)$-extension $\mathbb{A}$ if and only if*

$$\operatorname{Re} h = -m_\infty(-0).$$

In this case this unique $(*)$-extension has the form

$$\begin{aligned} \mathbb{A}y &= -y'' + q(x)y + [hy(a) - y'(a)]\,\delta(x-a), \\ \mathbb{A}^*y &= -y'' + q(x)y + [\overline{h}y(a) - y'(a)]\,\delta(x-a). \end{aligned} \tag{5.5}$$

Now we shall construct an L-system based on a non-self-adjoint Schrödinger operator. One can easily check that the $(*)$-extension

$$\mathbb{A}y = -y'' + q(x)y - \frac{1}{\mu - h}\,[y'(a) - hy(a)]\,[\mu\delta(x-a) + \delta'(x-a)], \quad \operatorname{Im} h > 0,$$

of the non-self-adjoint Schrödinger operator T_h of the form (5.2) satisfies the condition

$$\operatorname{Im} \mathbb{A} = \frac{\mathbb{A} - \mathbb{A}^*}{2i} = (.,g)g,$$

where

$$g = \frac{(\operatorname{Im} h)^{\frac{1}{2}}}{|\mu - h|}\,[\mu\delta(x-a) + \delta'(x-a)]$$

and $\delta(x-a), \delta'(x-a)$ are the delta-function and its derivative at the point a, respectively. Moreover,

$$(y, g) = \frac{(\operatorname{Im} h)^{\frac{1}{2}}}{|\mu - h|}\,[\mu y(a) - y'(a)],$$

where $y \in \mathcal{H}_+$, $g \in \mathcal{H}_-$, $\mathcal{H}_+ \subset L_2(a, +\infty) \subset \mathcal{H}_-$ and the triplet of Hilbert spaces is as discussed in Theorem 5.1. Let $E = \mathbb{C}$, $Kc = cg$ $(c \in \mathbb{C})$. It is clear that

$$K^*y = (y, g), \quad y \in \mathcal{H}_+, \tag{5.6}$$

and $\operatorname{Im} \mathbb{A} = KK^*$. Therefore, the array

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset L_2[a, +\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}, \tag{5.7}$$

is an L-system with the main operator $\mathbb{A}$ of the form (5.3), the direction operator $J = 1$, and the channel operator K of the form (5.6). Our next logical step is finding the transfer function of (5.7). It was shown in [4], [6] that

$$W_\Theta(\lambda) = \frac{\mu - h}{\mu - \overline{h}}\;\frac{m_\infty(\lambda) + \overline{h}}{m_\infty(\lambda) + h},$$

and

$$V_\Theta(\lambda) = \frac{(m_\infty(\lambda) + \mu)\,\mathrm{Im}\,h}{(\mu - \mathrm{Re}\,h)\,m_\infty(\lambda) + \mu\mathrm{Re}\,h - |h|^2}. \tag{5.8}$$

The following theorem can be found in [4].

Theorem 5.3 *Let Θ be an L-system of the form* (5.7)*, where $\mathbb{A}$ is a $(*)$-extension of the form* (5.3) *of the accretive Schrödinger operator T_h of the form* (5.2)*. Then its impedance function $V_\Theta(z)$ is a Stieltjes function if and only if*

$$\mathrm{Re}\,h \geq -m_\infty(-0) \quad \textit{and} \quad \mu \geq \frac{(\mathrm{Im}\,h)^2}{m_\infty(-0) + \mathrm{Re}\,h} + \mathrm{Re}\,h. \tag{5.9}$$

6 Sectorial Schrödinger L-systems

Let Θ be an L-system of the form (5.7), where $\mathbb{A}$ is a $(*)$-extension (5.3) of the accretive Schrödinger operator T_h. The following theorem [4] takes place.

Theorem 6.1 *If an accretive Schrödinger operator T_h, $(\mathrm{Im}\,h > 0)$ is α-sectorial, then*

$$\tan\alpha = \frac{\mathrm{Im}\,h}{\mathrm{Re}\,h + m_\infty(-0)}. \tag{6.1}$$

Conversely, if h, $(\mathrm{Im}\,h > 0)$ is such that $\mathrm{Re}\,h > -m_\infty(-0)$, then operator T_h of the form (5.2) *is α-sectorial and α is determined by* (6.1)*. Moreover, T_h is accretive but not α-sectorial for any $\alpha \in (0, \pi/2)$ if and only if* $\mathrm{Re}\,h = -m_\infty(-0)$.

It follows from Theorems 3.2 and 5.3 (see also [4]) that the operator $\mathbb{A}$ of Θ is accretive if and only if (5.9) holds. Using (5.8) we can write the impedance function $V_\Theta(z)$ in the form

$$V_\Theta(z) = \frac{(m_\infty(z) + \mu)\,\mathrm{Im}\,h}{(\mu - \mathrm{Re}\,h)\,(m_\infty(z) + \mathrm{Re}\,h) - (\mathrm{Im}\,h)^2}. \tag{6.2}$$

Consider our system Θ with $\mu = +\infty$. Then in (6.2) we obtain

$$V_\Theta(z) = \frac{\mathrm{Im}\,h}{m_\infty(z) + h}.$$

Thus, in this case

$$\lim_{x\to-\infty} V_\Theta(x) = \lim_{x\to-\infty} \frac{\mathrm{Im}\,h}{m_\infty(x) + h} = 0, \tag{6.3}$$

since $m_\infty(x) \to +\infty$ as $x \to -\infty$. Moreover,

$$\lim_{x\to-0} V_\Theta(x) = \frac{\mathrm{Im}\,h}{m_\infty(-0) + h}.$$

Assuming that T_h is α-sectorial and hence $\mathrm{Re}\,h > -m_\infty(-0)$, we use (4.2) and obtain

$$\lim_{x\to-\infty} V_\Theta(x) = 0 = \tan 0 = \tan\alpha_1, \quad \lim_{x\to-0} V_\Theta(x) = \frac{\mathrm{Im}\,h}{m_\infty(-0) + h} = \tan\alpha_2.$$

On the other hand since T_h is α-sectorial, then via Theorem 6.1 we have that

$$\tan\alpha = \tan\alpha_2 = \frac{\mathrm{Im}\,h}{m_\infty(-0) + h},$$

and hence, by Corollary 4.3, $V_\Theta(z)$ belongs to the class $S^{0,\alpha}$.

Let now $\mu \neq +\infty$ and satisfy the second inequality (5.9). Then

$$\lim_{x\to-\infty} V_\Theta(x) = \lim_{x\to-\infty} \frac{(m_\infty(x)+\mu)\operatorname{Im} h}{(\mu - \operatorname{Re} h)(m_\infty(x)+\operatorname{Re} h) - (\operatorname{Im} h)^2} = \frac{\operatorname{Im} h}{\mu - \operatorname{Re} h} = \tan\alpha_1, \tag{6.4}$$

and

$$\lim_{x\to-0} V_\Theta(x) = \frac{(m_\infty(-0)+\mu)\operatorname{Im} h}{(\mu - \operatorname{Re} h)(m_\infty(-0)+\operatorname{Re} h) - (\operatorname{Im} h)^2} = \tan\alpha_2. \tag{6.5}$$

Therefore, in this case $V_\Theta(z) \in S^{\alpha_1,\alpha_2}$.

Theorem 6.2 *Let Θ be an L-system of the form* (5.7)*, where $\mathbb{A}$ is a $(*)$-extension of an α-sectorial operator T_h with the exact angle of sectoriality $\alpha \in (0,\pi/2)$. Then $\mathbb{A}$ is an α-sectorial $(*)$-extension of T_h* (*with the same angle of sectoriality*) *if and only if $\mu = +\infty$ in* (5.3)*.*

P r o o f. It follows from (6.3)–(6.5) that in this case $V_\Theta(z) \in S^{0,\alpha}$ if and only if $\mu = +\infty$. Thus using Corollary 4.3 for the function $V_\Theta(z)$ we obtain that $\mathbb{A}$ is α-sectorial $(*)$-extension of T_h. □

We note that if T_h is α-sectorial with the exact angle of sectoriality α, then it admits only one α-sectorial $(*)$-extension $\mathbb{A}$ with the same angle of sectoriality α. Consequently, $\mu = +\infty$ and $\mathbb{A}$ has the form (5.5).

Theorem 6.3 *Let Θ be an L-system of the form* (5.7)*, where $\mathbb{A}$ is a $(*)$-extension of an α-sectorial operator T_h with the exact angle of sectoriality $\alpha \in (0,\pi/2)$. Then $\mathbb{A}$ is accretive but not α-sectorial for any $\alpha \in (0,\pi/2)$ $(*)$-extension of T_h if and only if in* (5.3)

$$\mu = \mu_0 = \frac{(\operatorname{Im} h)^2}{m_\infty(-0)+\operatorname{Re} h} + \operatorname{Re} h. \tag{6.6}$$

P r o o f. Let $V_\Theta(z)$ be the impedance function of our system Θ. If in (6.4) we set $\mu = \mu_0$ where μ_0 is given by (6.6), then

$$\lim_{x\to-\infty} V_\Theta(x) = \frac{\operatorname{Im} h}{\mu_0 - \operatorname{Re} h} = \frac{m_\infty(-0)+\operatorname{Re} h}{\operatorname{Im} h} = \frac{1}{\tan\alpha} = \tan\left(\frac{\pi}{2}-\alpha\right) = \tan\alpha_1, \tag{6.7}$$

where $\alpha_1 = \frac{\pi}{2} - \alpha$. On the other hand, using (6.5) with $\mu = \mu_0$ we obtain

$$\lim_{x\to-0} V_\Theta(x) = \frac{\operatorname{Im} h\left(m_\infty(-0) + \frac{(\operatorname{Im} h)^2}{m_\infty(-0)+\operatorname{Re} h}\right)}{\frac{(\operatorname{Im} h)^2}{m_\infty(-0)+\operatorname{Re} h}(m_\infty(-0)+\operatorname{Re} h) - (\operatorname{Im} h)^2} = \infty = \tan\frac{\pi}{2} = \tan\alpha_2. \tag{6.8}$$

Hence, (6.7) and (6.8) yield $V_\Theta(z) \in S^{\frac{\pi}{2}-\alpha,\frac{\pi}{2}}$. Now, if we assume the α-sectoriality of $\mathbb{A}$, then then by Theorem 4.2

$$\tan\alpha > \tan\alpha_2 = \infty.$$

Therefore, $\mathbb{A}$ is accretive but not α-sectorial for any $\alpha \in (0,\pi/2)$.

Conversely, suppose, that $\mathbb{A}$ is an α-sectorial $(*)$-extension for some $\alpha \in (0,\pi/2)$. Then, according to Theorem 4.5, $\mathbb{A}$ is also β-sectorial and

$$\tan\beta = \tan\alpha_2 + 2\sqrt{\tan\alpha_1(\tan\alpha_2 - \tan\alpha_1)} < \infty.$$

Hence, $\tan\alpha_2 \neq \infty$ and it follows from (6.8) that $\mu \neq \mu_0$. The theorem is proved. □

Note that it follows from the above theorem that any α-sectorial operator T_h with the exact angle of sectoriality $\alpha \in (0,\pi/2)$ admits only one accretive $(*)$-extension $\mathbb{A}$. This extension takes the form (5.3) with $\mu = \mu_0$ where μ_0 is given by (6.6).

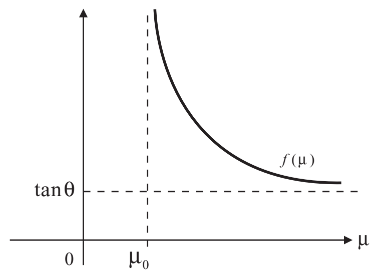

Fig. 1 Function $f(\mu)$.

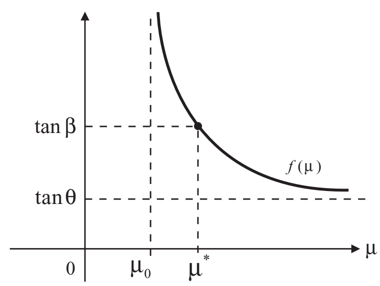

Fig. 2 Angle of sectoriality β.

Theorem 6.4 *Let Θ be an accretive L-system of the form* (5.7)*, where $\mathbb{A}$ is a $(*)$-extension of a θ-sectorial operator T_h. Let also $\mu_* \in (\mu_0, +\infty)$ be a fixed value that parameterizes $\mathbb{A}$ via* (5.3)*, μ_0 be defined by* (6.6)*, and $V_\Theta(z) \in S^{\alpha_1,\alpha_2}$. Then a $(*)$-extension $\mathbb{A}_\mu$ of T_h is β-sectorial for any $\mu \in [\mu_*, +\infty)$ with*

$$\tan\beta = \tan\alpha_1 + 2\sqrt{\tan\alpha_1\ \tan\alpha_2}. \tag{6.9}$$

P r o o f. According to Theorems 4.2 and 4.5, a φ-sectorial operator $\mathbb{A}$ of an L-system of the form (5.7) with the impedance function of the class S^{α_1,α_2} is also α-sectorial with

$$\tan\alpha = \tan\alpha_2 + 2\sqrt{\tan\alpha_1(\tan\alpha_2 - \tan\alpha_1)}.$$

But then, clearly

$$\tan\alpha < \tan\beta = \tan\alpha_1 + 2\sqrt{\tan\alpha_1\ \tan\alpha_2}, \tag{6.10}$$

and hence this $\mathbb{A}$ is also β-sectorial.

Now suppose $\mu \in (\mu_0, +\infty)$. Then it follows from Theorem 6.3 that the operator $\mathbb{A}$ in L-system Θ of the form (5.7) is φ-sectorial (with some angle φ) for any such μ in parametrization (5.3). Using (6.4) and (6.5) on the impedance function $V_\Theta(z)$ of this L-system we can define a function

$$\begin{aligned} f(\mu) &= \tan\alpha_1 + 2\sqrt{\tan\alpha_1\ \tan\alpha_2} \\ &= \frac{(m_\infty(x)+\mu)\,\mathrm{Im}\,h}{(\mu-\mathrm{Re}\,h)\,(m_\infty(x)+\mathrm{Re}\,h)-(\mathrm{Im}\,h)^2} \\ &\quad + 2\sqrt{\frac{\mathrm{Im}\,h}{\mu-\mathrm{Re}\,h}\cdot\frac{(m_\infty(x)+\mu)\,\mathrm{Im}\,h}{(\mu-\mathrm{Re}\,h)\,(m_\infty(x)+\mathrm{Re}\,h)-(\mathrm{Im}\,h)^2}}. \end{aligned} \tag{6.11}$$

By direct check one confirms that $f(\mu)$ is a decreasing function defined on $(\mu_0, +\infty)$ with the range $[\tan\theta, +\infty)$, where θ is the angle of sectoriality of the operator T_h and $\tan\theta$ is given by (6.1). The graph of this functions is schematically given on the Figure 1.

Next we take the $(*)$-extension $\mathbb{A}$ that is parameterized via (5.3) by the fixed value $\mu_* \in (\mu_0, +\infty)$ from the premise of our theorem. According to our derivations above this $\mathbb{A}$ is β-sectorial with β given by (6.9). But then for every $\mu \in (\mu_*, +\infty)$ the values of $f(\mu)$ are going to be smaller than $\tan\beta$ (see Figure 2). Consequently, for a $(*)$-extension $\mathbb{A}_\mu$ that is parameterized by the value of $\mu \in [\mu_*, +\infty)$ the following obvious inequalities take place

$$|\operatorname{Im}(\mathbb{A}_\mu f, f)| \le f(\mu) \operatorname{Re}(\mathbb{A}_\mu f, f) \le (\tan\beta) \operatorname{Re}(\mathbb{A}_\mu f, f), \quad f \in \mathcal{H}_+.$$

Hence, any $(*)$-extension $\mathbb{A}_\mu$ parameterized by a $\mu \in [\mu_*, +\infty)$ is β-sectorial. □

Note that Theorem 6.4 provides us with a value β which serves as a universal angle of sectoriality for the entire family of $(*)$-extensions $\mathbb{A}$ of the form (5.3). The next theorem provides us with the existence of a real number μ^* described in Theorem 6.4.

Theorem 6.5 *Let Θ be an L-system of the form* (5.7)*, where $\mathbb{A}$ is an α-sectorial $(*)$-extension of a θ-sectorial operator T_h and $V_\Theta(z) \in S^{\alpha_1,\alpha_2}$. Then there exists a real number μ^* that can be derived from equation* (6.9) *such that any $(*)$-extension $\mathbb{A}$ parameterized by a $\mu \in [\mu_*, +\infty)$ is a β-sectorial $(*)$-extension of T_h.*

The proof directly follows from Theorem 6.4.

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