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Multiplication Theorems for J-contractive Operator-valued Funcitons

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Abstract. Using methods developed in [**5**]–[**8**] we define classes of J-contractive in the half-plane operator-valued functions. A member of each class is shown to be realizable as a transfer mappings of systems θ with, generally speaking, unbounded main operator. A problem when the product of J-contractive operator-valued functions from selected class belongs to the same (or another) class is investigated.

1 Introduction

In this paper we continue the investigation of various problems that arise in the study of linear stationary conservative dynamic systems (operator colligations). Relying on the results and technique developed in [**6**],[**7**] we keep dealing with linear stationary conservative dynamic systems (l.s.c.d.s) θ of the form

$$\begin{cases} (\mathbb{A}-zI)=KJ\varphi_- \\ \varphi_+=\varphi_- -2iK^*x \end{cases} \qquad (\mathrm{Im}\ \mathbb{A}=KJK^*)$$

or

$$\theta=\begin{pmatrix} \mathbb{A} & K & J \\ \mathfrak{H}_+\subset\mathfrak{H}\subset\mathfrak{H}_- & & E \end{pmatrix}.$$

In the system θ above $\mathbb{A}$ is a bounded linear operator acting from $\mathfrak{H}_+$ into $\mathfrak{H}_-$, where $\mathfrak{H}_+\subset\mathfrak{H}\subset\mathfrak{H}_-$ is a rigged Hilbert space, $\mathbb{A}\supset T\supset A$, $\mathbb{A}^*\supset T^*\supset A$, A is a Hermitian operator in $\mathfrak{H}$, T is a non-Hermitian operator in $\mathfrak{H}$, K is a linear bounded operator from E into $\mathfrak{H}_-$, $J=J^*=J^{-1}$ is acting in E, $\varphi_\pm\in E$, φ_- is an input

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vector, φ_+ is an output vector, and $x \in \mathfrak{H}_+$ is a vector of the inner state of the system θ. The operator-valued function

$$W_\theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ \qquad (\varphi_+ = W_\theta(z)\varphi_-),$$

is the transfer operator-valued function of the system θ.

We consider class Ω of J-contractive in the lower half-plane operator-valued functions that can be realized directly as transfer fucntions of some systems θ described above (see also [**1**],[**35**]). A class Ω is partitioned into subclasses depending on the properties of operators in corresponding realizing systems. Three subclasses are described:

(1) a subclass for which $\overline{\mathfrak{D}(A)} = \mathfrak{H}$, $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$
(2) a subclass for which $\overline{\mathfrak{D}(A)} \neq \mathfrak{H}$, $\mathfrak{D}(T) = \mathfrak{D}(T^*)$
(3) a subclass for which $\overline{\mathfrak{D}(A)} \neq \mathfrak{H}$, $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$

A problem when the product of two operator-valued functions from selected subclass belongs to the same (or another) subclass is investigated. To prove the multiplication theorems in each subclass we generalize the concept of a product of two systems which generally speaking is an unbounded version of the Brodskiĭ-Livšic operator colligations product described in [**11**] (see also [**2**]). This approach allows us to constructively derive the realization of the product of two transfer operator-valued functions from the classes considered.

Theorem 5.3 is a version of the well-known Potapov-Ginzburg transformation. It establishes the relationship between contractive and J-contractive in the lower half-plane operator-valued functions from above mentioned subclasses.

Note that theorems 6.3 − 6.9 offer a further development and complement of the results by D.R. Arov, M.A. Nudelman [**4**], M.S. Brodskiĭ [**11**], A.V. Kuzhel [**21**], M.S. Livšic [**22**],[**23**], V.P. Potapov [**24**], A.V. Shtraus [**26**],[**27**], H. Bart, I. Gohberg, M. Kaashoek [**9**] (see also the survey [**13**]).

2 Preliminary Results

Let $\mathfrak{H}$ denote a Hilbert space with inner product (x, y) and let A be a closed linear Hermitian operator, i.e. $(Ax, y) = (x, Ay)$ $(\forall x, y \in \mathfrak{D}(A))$, acting on a Hilbert space $\mathfrak{H}$ with generally speaking, non-dense domain $\mathfrak{D}(A)$. Let $\mathfrak{H}_0 = \overline{\mathfrak{D}(A)}$ and A^* be the adjoint to the operator A (we consider A as acting from $\mathfrak{H}_0$ into $\mathfrak{H}$).

We denote $\mathfrak{H}_+ = \mathfrak{D}(A^*)$ $(\overline{\mathfrak{D}(A^*)} = \mathfrak{H})$ with inner product

$$(f, g)_+ = (f, g) + (A^*f, A^*g) \qquad (f, g \in \mathfrak{H}_+) \tag{2.1}$$

and then construct the *rigged* Hilbert space $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$ [**10**], [**8**]. Here $\mathfrak{H}_-$ is the space of all linear functionals over $\mathfrak{H}_+$ that are continuous with respect to $\|\cdot\|_+$. The norms of these spaces are connected by the relations $\|x\| \leq \|x\|_+$ $(x \in \mathfrak{H}_+)$, $\|x\|_- \leq \|x\|$ $(x \in \mathfrak{H})$. The Riesz-Berezanskii operator (see [**8**]) $\mathcal{R}$ maps $\mathfrak{H}_-$ onto $\mathfrak{H}_+$ such that

$$\begin{aligned} (x, y)_- = (x, \mathcal{R}y) = (\mathcal{R}x, y) = (\mathcal{R}x, \mathcal{R}y)_+ \qquad & (x, y \in \mathfrak{H}_-) \\ (u, v)_+ = (u, \mathcal{R}^{-1}v) = (\mathcal{R}^{-1}u, v) = (\mathcal{R}^{-1}x, \mathcal{R}^{-1}y)_- \qquad & (u, v \in \mathfrak{H}_+) \end{aligned} \tag{2.2}$$

In what follows we use symbols $(+)$, $(\cdot)$, and $(-)$ to indicate the norms $\|\cdot\|_+$, $\|\cdot\|$, and $\|\cdot\|_-$ by which geometrical and topological concepts are defined in $\mathfrak{H}_+$, $\mathfrak{H}$, and $\mathfrak{H}_-$, respectively.

In the above settings $\mathfrak{D}(A) \subset \mathfrak{D}(A^*)(= \mathfrak{H}_+)$ and $A^*y = PAy$ $(\forall y \in \mathfrak{D}(A))$, where P is an orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_0$. Let

$$\mathfrak{L} := \mathfrak{H} \ominus \mathfrak{H}_0 \quad \mathfrak{M}_\lambda := (A - \lambda I)\mathfrak{D}(A) \quad \mathfrak{N}_\lambda := (\mathfrak{M}_{\bar{\lambda}})^\perp. \tag{2.3}$$

The subspace $\mathfrak{N}_\lambda$ is called a *defect subspace* of A for the point $\bar{\lambda}$. The cardinal number $\dim\mathfrak{N}_\lambda$ remains constant when λ is in the upper half-plane. Similarly, the number $\dim\mathfrak{N}_\lambda$ remains constant when λ is in the lower half-plane. The numbers $\dim\mathfrak{N}_\lambda$ and $\dim\mathfrak{N}_{\bar{\lambda}}$ $(\mathrm{Im}\lambda < 0)$ are called the *defect numbers* or *deficiency indices* of operator A [**1**]. The subspace $\mathfrak{N}_\lambda$ which lies in $\mathfrak{H}_+$ is the set of solutions of the equation $A^*g = \lambda Pg$.

Let now P_λ be the orthogonal projection onto $\mathfrak{N}_\lambda$, set

$$\mathfrak{B}_\lambda = P_\lambda \mathfrak{L}, \qquad \mathfrak{N}'_\lambda = \mathfrak{N}_\lambda \ominus \overline{\mathfrak{B}_\lambda} \tag{2.4}$$

It is easy to see that $\mathfrak{N}'_\lambda = \mathfrak{N}_\lambda \cap \mathfrak{H}_0$ and $\mathfrak{N}'_\lambda$ is the set of solutions of the equation $A^*g = \lambda g$ (see [**36**]).

The subspace $\mathfrak{N}'_\lambda$ is the defect subspace of the densely defined Hermitian operator PA on $\mathfrak{H}_0$ (see [**33**]). The numbers $\dim\mathfrak{N}'_\lambda$ and $\dim\mathfrak{N}'_{\bar{\lambda}}$ $(\mathrm{Im}\lambda < 0)$ are called *semi-defect numbers* or the *semi-deficiency indices* of the operator A [**22**]. The von Neumann formula

$$\mathfrak{H}_+ = \mathfrak{D}(A^*) = \mathfrak{D}(A) + \mathfrak{N}_\lambda + \mathfrak{N}_{\bar{\lambda}}, \qquad (\mathrm{Im}\lambda \neq 0), \tag{2.5}$$

holds, but this decomposition is not direct for a non-densely defined operator A. There exists a generalization of von Neumann's formula [**1**],[**35**] to the case of a non-densely defined Hermitian operator (direct decomposition). We call an operator A *regular*, if PA is a closed operator in $\mathfrak{H}_0$. For a regular operator A we have

$$\mathfrak{H}_+ = \mathfrak{D}(A) + \mathfrak{N}'_\lambda + \mathfrak{N}'_{\bar{\lambda}} + \mathfrak{N}, \qquad (\mathrm{Im}\lambda \neq 0) \tag{2.6}$$

where $\mathfrak{N} := \mathcal{R}\mathfrak{L}$, $\mathcal{R}$ is the Riesz-Berezanskii operator. This is a generalization of von Neumann's formula. For $\lambda = \pm i$ we obtain the $(+)$-orthogonal decomposition

$$\mathfrak{H}_+ = \mathfrak{D}(A) \oplus \mathfrak{N}'_i \oplus \mathfrak{N}'_{-i} \oplus \mathfrak{N}. \tag{2.7}$$

Let $\tilde{A}$ be a closed Hermitian extension of the operator A. Then $\mathfrak{D}(\tilde{A}) \subset \mathfrak{H}_+$ and $P\tilde{A}x = A^*x$ $(\forall x \in \mathfrak{D}(\tilde{A}))$. According to [**36**] a closed Hermitian extension $\tilde{A}$ is said to be *regular* if $P\tilde{A}$ is closed. This implies that $\mathfrak{D}(\tilde{A})$ is $(+)$-closed. According to the theory of extensions of closed Hermitian operators A with non-dense domain [**20**], an operator U $(\mathfrak{D}(U) \subseteq \mathfrak{N}_i$, $\mathfrak{R}(U) \subseteq \mathfrak{N}_{-i})$ is called an *admissible operator* if $(U - I)f_i \in \mathfrak{D}(A)$ $(f_i \in \mathfrak{D}(U))$ only for $f_i = 0$. Then (see [**3**]) any symmetric extension $\tilde{A}$ of the non-densely defined closed Hermitian operator A, is defined by an isometric admissible operator U, $\mathfrak{D}(U) \subseteq \mathfrak{N}_i$, $\mathfrak{R}(U) \subseteq \mathfrak{N}_{-i}$ by the formula

$$\tilde{A}f_{\tilde{A}} = Af_A + (-if_i - iUf_i), \quad f_A \in \mathfrak{D}(A) \tag{2.8}$$

where $\mathfrak{D}(\tilde{A}) = \mathfrak{D}(A) \dotplus (U - I)\mathfrak{D}(U)$. The operator $\tilde{A}$ is self-adjoint if and only if $\mathfrak{D}(U) = \mathfrak{N}_i$ and $\mathfrak{R}(U) = \mathfrak{N}_{-i}$.

A regular operator A is called *O-operator* if its semidefect numbers (defect numbers of an operator PA) are equal to zero.

Denote by $[\mathfrak{H}_1, \mathfrak{H}_2]$ the set of all linear bounded operators acting from Hilbert space $\mathfrak{H}_1$ into a Hilbert space $\mathfrak{H}_2$.

Definition 2.1 An operator $\mathbb{A} \in [\mathfrak{H}_+, \mathfrak{H}_-]$ is a *bi-extension* of A if both $\mathbb{A} \supset A$ and $\mathbb{A}^* \supset A$ hold.

If $\mathbb{A}=\mathbb{A}^*$, then $\mathbb{A}$ is called self-adjoint bi-extension of the operator A. We write $\mathfrak{S}(A)$ for the class of bi-extensions of A. This class is closed in the weak topology and is invariant under taking adjoints (see [**5**], [**36**]).

Let $\mathbb{A}$ be a bi-extension of Hermitian operator A. The operator $\hat{A}f=\mathbb{A}f$, $\mathfrak{D}(\hat{A})=\{f\in\mathfrak{H},\mathbb{A}f\in\mathfrak{H}\}$ is called the *quasikernel* of $\mathbb{A}$. If $\mathbb{A}=\mathbb{A}^*$ and $\hat{A}$ is a quasi-kernel of $\mathbb{A}$ such that $A\neq\hat{A}$, $\hat{A}^*=\hat{A}$ then $\mathbb{A}$ is said to be a *strong* self-adjoint bi-extension of A.

Definition 2.2 We say that a closed densely defined linear operator T acting on a Hilbert space $\mathfrak{H}$ belongs to the class Ω_A if:

(1) $T\supset A$, $T^*\supset A$ where A is a closed Hermitian operator;
(2) $(-i)$ is a regular point of T. [1]

It was mentioned in [**5**] that lineals $\mathfrak{D}(T)$ and $\mathfrak{D}(T^*)$ are $(+)$-closed, the operators T and T^* are $(+,\cdot)$-bounded.

Definition 2.3 An operator $\mathbb{A}$ in $[\mathfrak{H}_+,\mathfrak{H}_-]$ is called a $(*)$-extension of an operator T of the class Ω_A if both $\mathbb{A}\supset T$ and $\mathbb{A}^*\supset T^*$.

This $(*)$-extension is called *correct*, if an operator $\mathbb{A}_R=\frac{1}{2}(\mathbb{A}+\mathbb{A}^*)$ is a strong self-adjoint bi-extension of an operator A. It is easy to show that if $\mathbb{A}$ is a $(*)$-extension of T, the T and T^* are quasi-kernels of $\mathbb{A}$ and $\mathbb{A}^*$, respectively.

Definition 2.4 We say the operator T of the class Ω_A belongs to the class Λ_A if

(1) T admits a correct $(*)$-extension;
(2) A is a maximal common Hermitian part of T and T^*.

It is known [**3**],[**36**] that if an operator T belongs to the class Ω_A and operator A is a maximal common Hermitian part of T and T^* that has finite and equal defect indices then T belongs to the class Λ_A.

The following theorem is referred to [**1**].

Theorem 2.5 *Let $\mathbb{A}$ be a self-adjoint bi-extension of Hermitian operator A. The necessary and sufficient condition for operator $(\mathbb{A}-\lambda I)^{-1}$ to be $(-,-)$-continuous is to be $(-,\cdot)$-continuous.*

Corollary 2.6 Let $\mathbb{A}$ be self-adjoint bi-extension of Hermitian operator A with finite and equal semi-deficiency indices. If for some λ operator $(\mathbb{A}-\lambda I)^{-1}$ is $(-,-)$-continuous then $\mathbb{A}$ is a strong bi-extension of operator A.

3 Linear Stationary Conservative Dynamic Systems

In this section we consider linear stationary conservative dynamic systems (l. s. c. d. s.) θ of the form

$$\begin{cases} (\mathbb{A}-zI)=KJ\varphi_- \\ \varphi_+=\varphi_--2iK^*x \end{cases} \qquad (\mathrm{Im}\ \mathbb{A}=KJK^*)\,. \tag{3.1}$$

In a system θ of the form (3.1) $\mathbb{A}$, K and J are bounded linear operators in Hilbert spaces, φ_- is an input vector, φ_+ is an output vector, x is an inner state vector of the system θ. For our purposes we need the following more precise definition:

[1]The condition, that $(-i)$ is a regular point in the definition of the class Ω_A is not essential. It is sufficient to require the existence of some regular point for T.

Definition 3.1 The array

$$\theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & E \end{pmatrix} \tag{3.2}$$

is called a linear stationary conservative dynamic system (l.s.c.d.s.) or Brodskiĭ-Livšic rigged operator colligation if

(1) $\mathbb{A}$ is a correct $(*)$-extension of an operator T of the class Λ_A.
(2) $J = J^* = J^{-1} \in [E, E], \quad \dim E < \infty$
(3) $\mathbb{A} - \mathbb{A}^* = 2iKJK^*$, where $K \in [E, \mathfrak{H}_-] \qquad (K^* \in [\mathfrak{H}_+, E])$

In this case, the operator K is called a *channel operator* and J is called a *direction operator* [**11**], [**23**]. A system θ of the form (3.2) is called a *scattering* system (*dissipative* operator colligation) if $J = I$. It can be shown [**11**] that if operator A of the system (3.2) has finite and equal deficiency indices then for any bi-extension $\mathbb{A}$ the direction operator K of the system θ can be chosen invertible. Therefore without loss of generality we will consider systems with invertible direction operators only.

We associate with the system θ an operator-valued function

$$W_\theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ \tag{3.3}$$

which is called *a transfer operator-valued function* of the system θ or a characteristic operator-valued function of Brodskiĭ-Livšic rigged operator colligations. It may be shown [**11**], that the transfer operator-function of the system θ of the form (3.2) has the following properties:

$$\begin{aligned} W_\theta^*(z)JW_\theta(z) - J &\geq 0 \quad (\mathrm{Im}\ z > 0, z \in \rho(T)) \\ W_\theta^*(z)JW_\theta(z) - J &= 0 \quad (\mathrm{Im}\ z = 0, z \in \rho(T)) \\ W_\theta^*(z)JW_\theta(z) - J &\leq 0 \quad (\mathrm{Im}\ z < 0, z \in \rho(T)) \end{aligned} \tag{3.4}$$

where $\rho(T)$ is the set of regular points of an operator T. Similar relations take place if we change $W_\theta(z)$ to $W_\theta^*(z)$ in (3.4). Thus, a transfer operator-valued function of the system θ of the form (3.2) is J-contractive in the lower half-plane on the set of regular points of an operator T and J-unitary on real regular points of an operator T.

Let θ be a l.s.c.d.s. of the form (3.2). We consider an operator-valued function

$$V_\theta(z) = K^*(\mathbb{A}_R - zI)^{-1}K. \tag{3.5}$$

The transfer operator-function $W_\theta(z)$ of the system θ and an operator-function $V_\theta(z)$ of the form (3.5) are connected by the relation

$$V_\theta(z) = i[W_\theta(z) + I]^{-1}[W_\theta(z) - I]J \tag{3.6}$$

4 Classes of Realizable Operator-Valued R-functions

As it is known [**29**] an operator-function $V(z) \in [E, E]$ is called an *operator-valued R-function* if it is holomorphic in the upper half-plane and $\mathrm{Im}\ V(z) \geq 0$ when $\mathrm{Im}\ z > 0$.

An operator-valued R-function, acting in Hilbert space $E (\dim E < \infty)$ has, as it is known [**19**],[**29**], integral representation

$$V(z) = Q + F \cdot z + \int_{-\infty}^{+\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2}\right) dG(t) \tag{4.1}$$

where $Q = Q^*$, $F \geq 0$ in the Hilbert space E, $G(t)$ is non-decreasing operator-function on $(-\infty, +\infty)$ for which

$$\int_{-\infty}^{+\infty} \frac{(dG(t)e, e)_E}{1+t^2} < \infty.$$

Definition 4.1 We call an operator-valued R-function acting in Hilbert space E $(\dim E < \infty)$ realizable if in some neighborhood of point $(-i)$ $V(z)$ can be represented in the form

$$V(z) = i[W_\theta(z) + I]^{-1}[W_\theta(z) - I]J \tag{4.2}$$

where $W_\theta(z)$ is a transfer operator-function of some l.s.c.d.s. θ with the direction operator J $(J = J^* = J^{-1} \in [E, E])$.

Definition 4.2 An operator-valued R-function $V(z) \in [E, E]$ $(\dim E < \infty)$ will be said to be a member of the class $N(R)$ if in the representation 4.1 we have

$$\begin{aligned} &i) \qquad F = 0, \\ &ii) \qquad Qe = \int_{-\infty}^{+\infty} \frac{t}{1+t^2} dG(t)e \end{aligned}$$

for all $e \in E$ such that

$$\int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty$$

The following result is found in [**7**]. Its proof is based on the relations (2.6)–(2.8).

Theorem 4.3 *Let θ be a l.s.c.d.s. of the form* (3.2), $\dim E < \infty$. *Then operator-function $V_\theta(z)$ of the form* 3.5, 3.6 *belongs to the class $N(R)$.*

Conversely, suppose that the operator-valued function $V(z)$ is acting on a finite-dimensional Hilbert space E and belong to the class $N(R)$. Then $V(z)$ admits a realization by the system θ of the form (3.2) *with a preassigned direction operator J for which $I + iV(-i)J$ is invertible.*

Remark 4.4 It was mentioned in [**7**] that when $J = I$ the invertibility condition for $I + iV(\lambda)J$ is satisfied automatically.

Now we are going to introduce three distinct subclasses of the class of realizable operator-valued functions $N(R)$.

Definition 4.5 An operator-valued R-function $V(z) \in [E, E]$ $(\dim E < \infty)$ of the class $N(R)$ is said to be a member of the subclass $N_0(R)$ if in the representation (4.1)

$$\int_{-\infty}^{+\infty} (dG(t)e, e)_E = \infty, \qquad (e \in E, e \neq 0).$$

Consequently, the operator-function $V(z)$ of the class $N_0(R)$ has a representation

$$V(z) = Q + \int_{-\infty}^{+\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) dG(t), \qquad (Q = Q^*). \tag{4.3}$$

Note, that the operator Q can be an arbitrary self-adjoint operator on the Hilbert space E.

Definition 4.6 An operator-valued R-function $V(z) \in [E, E] \quad (\dim E < \infty)$ of the class $N(R)$ is said to be a member of the subclass $N_1(R)$ if in the representation (4.1)

$$\int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty, \qquad (e \in E). \tag{4.4}$$

It is easy to see that operator-function $V(z)$ of the class $N_1(R)$ has a representation

$$V(z) = \int_{-\infty}^{+\infty} \frac{1}{t - z}\, dG(t). \tag{4.5}$$

Definition 4.7 An operator-valued R-function $V(z) \in [E, E], \qquad (\dim E < \infty)$ of the class $N(R)$ is said to be a member of the subclass $N_{01}(R)$ if the subspace

$$E_\infty = \left\{ e \in E : \int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty \right\}$$

possesses a property: $E_\infty \neq \emptyset, \quad E_\infty \neq E$.

One may notice that $N(R)$ is a union of three distinct subclasses $N_0(R)$, $N_1(R)$ and $N_{01}(R)$. The following theorem is an analogue of the Theorem 4.3 for the class $N_0(R)$.

Theorem 4.8 *Let θ be a l. s. c. d. s. of the form* (3.2), $\dim E < \infty$ *where A is a linear closed Hermitian operator with dense domain and $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$. Then operator-valued function $V_\theta(z)$ of the form* (3.5), (3.6) *belongs to the class $N_0(R)$.*

Conversely, let an operator-valued function $V(z)$ acting on a finite-dimensional Hilbert space E belong to the class $N_0(R)$. Then it admits a realization by the system θ of the form (3.2) *with a preassigned directional operator J for which $I + iV(-i)J$ is invertible, densely defined closed Hermitian operator A, and $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$.*

Analogous result for the class $N_1(R)$ is the contents of the following theorem [**8**].

Theorem 4.9 *Let θ be a l.s.c.d. s. of the form* (3.2), $\dim E < \infty$ *where A is a linear closed Hermitian O-operator and $\mathfrak{D}(T) = \mathfrak{D}(T^*)$. Then operator-valued function $V_\theta(\lambda)$ of the form* (3.5), (3.6) *belongs to the class $N_1(R)$.*

Conversely, suppose that an operator-valued function $V(z)$ is acting on a finite-dimensional Hilbert space E and belongs to the class $N_1(R)$. Then it admits a realization by the system θ of the form (3.2) *with a preassigned directional operator J for which $I + iV(-i)J$ is invertible, a linear closed regular Hermitian O-operator A with a non-dense domain, and $\mathfrak{D}(T) = \mathfrak{D}(T^*)$.*

The following theorem [**8**] completes the section by establishing direct and inverse realization results for the remaining subclass of realizable operator-valued R-functions $N_{01}(R)$.

Theorem 4.10 *Let θ be a l. s. c. d. s. of the form* (3.2), $\dim E < \infty$ *where A is a linear closed Hermitian operator with non-dense domain and $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$. Then operator-valued function $V_\theta(z)$ of the form* (3.5), (3.6) *belongs to the class $N_{01}(R)$.*

Conversely, suppose that an operator-valued function $V(z)$ is acting on a finite-dimensional Hilbert space E and belongs to the class $N_{01}(R)$. Then it admits a realization by the system θ of the form (3.2) *with a preassigned directional operator*

J for which $I+iV(-i)J$ is invertible, a linear closed regular Hermitian operator A with a non-dense domain, and $\mathfrak{D}(T)\neq\mathfrak{D}(T^*)$.

5 Classes $\Omega(R,J)$. Potapov-Ginzburg Transformation

In this section we introduce a class of J-contractive in a half-plane operator-valued functions $\Omega(R,J)$.

Definition 5.1 An operator-valued function $W(z)$ acting in finite-dimensional Hilbert space E is said to be a member of the class $\Omega(R,J)$ ($J=J^*=J^{-1}$) if there exists a l.s.c.d.s.

$$\theta=\begin{pmatrix} \mathbb{A} & K & J\\ \mathfrak{H}_+\subset\mathfrak{H}\subset\mathfrak{H}_- & & E\end{pmatrix}$$

such that

$$W(z)=W_\theta(z)=I-2iK^*(\mathbb{A}-zI)^{-1}KJ$$

in some neighborhood of $(-i)$.

The relations

$$\begin{aligned} W^*(z)JW(z)-J\geq 0 \quad &(\mathrm{Im}\ z>0, z\in\rho(T))\\ W^*(z)JW(z)-J= 0 \quad &(\mathrm{Im}\ z=0, z\in\rho(T))\\ W^*(z)JW(z)-J\leq 0 \quad &(\mathrm{Im}\ z<0, z\in\rho(T))\end{aligned}$$

therefore hold true for all the functions of $\Omega(R,J)$ class. Thus, $\Omega(R,J)$ consists of J-contractive in a lower half-plane functions. The definition also implies that $W(z)$ belongs to $\Omega(R,J)$ if and only if it is holomorphic in some neighborhood of $(-i)$ and operator-valued function

$$V(z)=i[W(z)+I]^{-1}[W(z)-I]J \tag{5.1}$$

belongs to the class $N(R)$ defined in the previous section. Taking this into account we introduce the following definition.

Definition 5.2 An operator-valued function $W(z)$ of the class $\Omega(R,J)$ belongs to the class $\Omega_0(R,J)$ (resp. $\Omega_1(R,J)$, $\Omega_{01}(R,J)$) if it is holomorphic in some neighborhood of $(-i)$ and operator-valued function $V(z)$ defined by (5.1) belongs to the class $N_0(R)$ (resp. $N_1(R,J)$, $N_{01}(R,J)$).

The theorem below is a version of Potapov-Ginzburg transformation. Together with the corollary 5.4 it establishes the relation between contractive and J-contractive in the half-plane operator-valued functions from the classes $\Omega(R,J)$, $\Omega_0(R,J)$, $\Omega_1(R,J)$, and $\Omega_{01}(R,J)$.

Theorem 5.3 *Let operator-valued function $W(z)$ belong to the class $\Omega(R,J)$. Let also P^+ and P^- be a pair of orthoprojections of the form*

$$P^+=\frac{1}{2}(I+J)\qquad \textit{and}\qquad P^-=\frac{1}{2}(I-J)$$

Then there exists an operator-function $\Sigma(z)$ of $\Omega(R,I)$ class such that

$$W(z)=(P^+\Sigma(z)-P^-)(P^+-P^-\Sigma(z))^{-1}.$$

Proof Let

$$V(z)=i[W(z)+I]^{-1}[W(z)-I]J.$$

Since $W(z)$ belongs to $\Omega(R,J)$ we have that $V(z)$ belongs to the class $N(R)$ and by the Theorem 4.3 can be realized by the *scattering* system

$$\theta' = \begin{pmatrix} \mathbb{A}' & K' & I \\ \mathfrak{H}'_+ \subset \mathfrak{H}' \subset \mathfrak{H}'_- & & E \end{pmatrix}.$$

Latter implies that

$$\begin{aligned} V(z) = V_{\theta'}(z) &= K'^*(\mathbb{A}'_R - zI)^{-1}K' \\ &= i[W_{\theta'}(z)+I]^{-1}[W_{\theta'}(z)-I] \end{aligned}$$

in some neighborhood of the point $(-i)$ where

$$W_{\theta'}(z) = I - 2iK'^*(\mathbb{A}' - zI)^{-1}K'.$$

It is clear that $W_{\theta'}(z)$ belongs to the $\Omega(R,I)$ class. Therefore

$$i[W(z)+I]^{-1}[W(z)-I]J = i[W_{\theta'}(z)+I]^{-1}[W_{\theta'}(z)-I]$$

where $W(z) \in \Omega(R,J)$ and $W_{\theta'}(z) \in \Omega(R,I)$. The above statement takes place in some neighborhood of $(-i)$.

Now let $\Sigma(z) \equiv W_{\theta'}(z)$. Then

$$\begin{aligned} (W(z)+I)^{-1}(W(z)-I)J &= [\Sigma(z)+I]^{-1}[\Sigma(z)-I] \\ &= [\Sigma(z)-I][\Sigma(z)+I]^{-1}, \end{aligned}$$

Multiplication by $[W(z)+I]$ from the left and by $[\Sigma(z)+I]^{-1}$ produces

$$[W(z)-I]J[\Sigma(z)+I] = [W(z)+I][\Sigma(z)-I].$$

Taking into account that $P^+ - P^- = J$ and $P^+ + P^- = I$ we obtain

$$[W(z)-I](P^+ - P^-)[\Sigma(z)+I] = [W(z)+I]J[\Sigma(z)-I],$$

or

$$\begin{aligned} W(z)P^+\Sigma(z) - W(z)P^-\Sigma(z) - P^+\Sigma(z) + P^-\Sigma(z) \\ + W(z)P^+ - W(z)P^- - P^+ + P^- \\ = W(z)\Sigma(z) - W(z) + \Sigma(z) - I, \end{aligned}$$

$$W(z)[P^+\Sigma(z) - P^-\Sigma(z) + 2P^+ - \Sigma(z)] = [P^+\Sigma(z) - P^-\Sigma(z) + \Sigma(z) - 2P^-],$$

or

$$W(z)[2P^+ - 2P^-\Sigma(z)] = [2P^+\Sigma(z) - 2P^-].$$

Cancelling yields

$$W(z)[P^+ - P^-\Sigma(z)] = [P^+\Sigma(z) - P^-]. \tag{5.2}$$

Let us show that operator $[P^+ - P^-\Sigma(z)]$ is invertible. We choose $x \in E$ such that

$$[P^+ - P^-\Sigma(z)]x = 0. \tag{5.3}$$

Then (5.2) implies

$$[P^+\Sigma(z) - P^-]x = 0. \tag{5.4}$$

We apply $P+$ to both sides of (5.3) and obtain

$$P^+[P^+ - P^-\Sigma(z)]x = 0,$$

or $P^+x = 0$. Similarly, we apply P^- to both sides of (5.4) and get that $P^-x = 0$. Thus $x = 0$ and operator $[P^+ - P^-\Sigma(z)]$ is invertible. Using this we obtain

$$W(z) = [P^+\Sigma(z) - P^-][P^+ - P^-\Sigma(z)]^{-1}, \tag{5.5}$$

that proves the theorem. □

Corollary 5.4 Let operator valued function $W(z)$ belong to the class $\Omega_0(R,J)$ (resp. $\Omega_1(R,J)$, $\Omega_{01}(R,J)$), $P^+=1/2(I+J)$, $P^-=1/2(I-J)$. Then there exists an operator-valued function $\Sigma(z)$ from $\Omega_0(R,J)$ (resp. $\Omega_1(R,J)$, $\Omega_{01}(R,J)$) class such that
$$W(z)=[P^+\Sigma(z)-P^-][P^+-P^-\Sigma(z)]^{-1}.$$

The Corollary 5.4 is proved in exactly the same way the Theorem 5.3 is.

6 Multiplication Theorems for $\Omega(R,J)$ classes

In this section we state and prove multiplication theorems for the operator-valued functions of $\Omega(R,J)$ class.

Definition 6.1 Two systems
$$\theta_1=\begin{pmatrix} \mathbb{A}_1 & K_1 & J_1\\ \mathfrak{H}_{+1}\subset\mathfrak{H}_1\subset\mathfrak{H}_{-1} & & E_1\end{pmatrix}\quad\text{and}\quad \theta_2=\begin{pmatrix} \mathbb{A}_2 & K_2 & J_2\\ \mathfrak{H}_{+2}\subset\mathfrak{H}_2\subset\mathfrak{H}_{-2} & & E_2\end{pmatrix}$$
are equal if and only if $\mathfrak{H}_{+1}=\mathfrak{H}_{+2}$, $\mathfrak{H}_1=\mathfrak{H}_2$, $\mathfrak{H}_{-1}=\mathfrak{H}_{-2}$, $\mathbb{A}_1=\mathbb{A}_2$, $K_1=K_2$, $J_1=J_2$, $E_1=E_2$.

Let θ_1 and θ_2 be two systems defined as above. Let
$$\mathfrak{H}_+=\mathfrak{H}_{+1}\oplus\mathfrak{H}_{+2},\quad \mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2,\quad \mathfrak{H}_-=\mathfrak{H}_{-1}\oplus\mathfrak{H}_{-2},$$
and $P_k:\mathfrak{H}\to\mathfrak{H}_k$, $P_k^+:\mathfrak{H}_+\to\mathfrak{H}_{+k}$, and $P_k^-:\mathfrak{H}_-\to\mathfrak{H}_{-k}$ $(k=1,2)$ denote the set of orthoprojections.

In the space $\mathfrak{H}$ we introduce an operator
$$\tilde A=A_1\oplus A_2, \tag{6.1}$$
where $A_1\subset T_1\subset\mathbb{A}_1$, $A_2\subset T_2\subset\mathbb{A}_2$ are correspondent elements of θ_1 and θ_2, respectively. Moreover, $\mathfrak{H}_{+1}=\mathfrak{D}(A_1^*)$ and $\mathfrak{H}_{+2}=\mathfrak{D}(A_2^*)$. Consequently,
$$\tilde A^*=A_1^*\oplus A_2^*, \tag{6.2}$$
and $\mathfrak{H}_+=\mathfrak{D}(\tilde A^*)=\mathfrak{D}(A_1^*)\oplus\mathfrak{D}(A_2^*)$.

The formulas below define operators $\tilde{\mathbb{A}}:\mathfrak{H}_+\to\mathfrak{H}_-$ and $\tilde{\mathbb{A}}^*:\mathfrak{H}_+\to\mathfrak{H}_-$ as
$$\begin{aligned}\tilde{\mathbb{A}}&=\mathbb{A}_1P_1^++\mathbb{A}_2P_2^++2iK_1JK_2^*P_2^+,\\ \tilde{\mathbb{A}}^*&=\mathbb{A}_1^*P_1^++\mathbb{A}_2^*P_2^++2iK_1JK_2^*P_2^+,\end{aligned} \tag{6.3}$$
Let also
$$\begin{aligned}\mathfrak{D}(T)&=\big\{x\in\mathfrak{H}_+:\ \tilde{\mathbb{A}}x\in\mathfrak{H}\big\},\\ \mathfrak{D}(T^*)&=\big\{x\in\mathfrak{H}_+:\ \tilde{\mathbb{A}}^*x\in\mathfrak{H}\big\}.\end{aligned} \tag{6.4}$$

We define operators T and T^* on these sets:
$$\begin{aligned}Tx&=\tilde{\mathbb{A}}x,\quad x\in\mathfrak{D}(T),\\ T^*x&=\tilde{\mathbb{A}}^*x,\quad x\in\mathfrak{D}(T^*),\end{aligned} \tag{6.5}$$
Operators $\tilde K:\,E\to\mathfrak{H}_-$ and $\tilde K^*:\mathfrak{H}_+\to E$ are defined in the following way
$$\tilde K=K_1+K_2, \tag{6.6}$$
$$\tilde K^*=K_1^*P_1^++K_2^*P_2^+. \tag{6.7}$$

We can show now that $T \supset \tilde{A}$ and $T^* \supset \tilde{A}$. Indeed, let $x = x_1 + x_2$ is an element of $\mathfrak{D}(\tilde{A}) = \mathfrak{D}(A_1) \oplus \mathfrak{D}(A_2)$. Then

$$\tilde{\mathbb{A}}x = \mathbb{A}_1 x_1 + \mathbb{A}_2 x_2 + 2iK_1JK_2^* x_2 = A_1 x_1 + A_2 x_2.$$

Right hand side of the latter expression belongs to the space $\mathfrak{H}$ since $K_1JK_2^*x_2 = 0$ due to the invertability of K_2 and the fact that

$$K_2JK_2^*x_2 = \mathrm{Im}\mathbb{A}_2 x_2 = 0, \qquad x \in \mathfrak{D}(A_2).$$

Hence, $T \supset \tilde{A}$. Similarly, one can show that $T^* \supset \tilde{A}$.

Let us set

$$\mathfrak{D}(A) = \{x \in \mathfrak{H}_+ : Tx = T^*x\}$$

and define an operator A on $\mathfrak{D}(A)$

$$Ax = \tilde{\mathbb{A}}x, \qquad x \in \mathfrak{D}(A). \tag{6.8}$$

We show that $A \supset \tilde{A}$. Let us pick an element x_A from $\mathfrak{D}(A)$ such that $x_1 = P_1^+ x_A$ and $x_2 = P_2^+ x_A$. Then the following holds

$$\mathbb{A}_1 x_1 + \mathbb{A}_2 x_2 + 2iK_1JK_2^* = \mathbb{A}_1^* x_1 + \mathbb{A}_2^* x_2 - 2iK_2JK_1^* x_1 \in \mathfrak{H}.$$

Taking into account $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ we make a projection of the last equality onto $\mathfrak{H}_1$ and $\mathfrak{H}_2$ we obtain

$$\mathbb{A}_1 x_1 + 2iK_1JK_2^* x_2 = \mathbb{A}_1^* x_1 \in \mathfrak{H}_1, \tag{6.9}$$

and

$$\mathbb{A}_2 x_2 = \mathbb{A}_2^* x_2 - 2iK_2JK_1^* x_1 \in \mathfrak{H}_2. \tag{6.10}$$

This immediately implies that $x_1 \in \mathfrak{D}(T_1^*)$ and $x_2 \in \mathfrak{D}(T_2)$. Moreover, first equation yields

$$\begin{aligned}
\mathbb{A}_1 x_1 + 2iK_1JK_2^* x_2 &= \mathbb{A}_1^* x_1,\\
\mathbb{A}_1 x_1 - \mathbb{A}_1^* x_1 &= -2iK_1JK_2^* x_2,\\
\frac{1}{2i}(\mathbb{A}_1 - \mathbb{A}_1^*)x_1 &= -K_1JK_2^* x_2,\\
K_1JK_1^* x_1 &= -K_1JK_2^* x_2,\\
K_1JK_1^* x_1 + K_1JK_2^* x_2 &= 0,\\
K_1J(K_1^* x_1 + K_2^* x_2) &= 0.
\end{aligned}$$

Since operator K_1 is invertible

$$K_1^* x_1 + K_2^* x_2 = 0, \tag{6.11}$$

or

$$K_1^* x_1 = -K_2^* x_2.$$

Let us show now that if $x_1 \in \mathfrak{D}(T_1^*)$, $x_2 \in \mathfrak{D}(T_2)$ and $K_1^* x_1 + K_2^* x_2 = 0$ then $x = x_1 + x_2$ belongs to $\mathfrak{D}(A)$.

$$\begin{aligned}
\tilde{\mathbb{A}}x &= \mathbb{A}_1 x_1 + \mathbb{A}_2 x_2 + 2iK_1JK_2^* x_2\\
&= \mathbb{A}_1 x_1 + T_2 x_2 - 2iK_1JK_1^* x_1\\
&= \mathbb{A}_1 x_1 + T_2 x_2 - \mathbb{A}_1 x_1 + \mathbb{A}_1^* x_1\\
&= T_2 x_2 + T^* x_1 \in \mathfrak{H}.
\end{aligned}$$

Therefore, $\tilde{\mathbb{A}}x = \tilde{\mathbb{A}}^* x$ belongs to $\mathfrak{H}$ or

$$Tx = T^*x,$$

that implies that $x \in \mathfrak{D}(A)$. It is easy to see now that

$$\mathfrak{D}(A) = \{x \in \mathfrak{H}_+ : x = x_1 + x_2, x_1 \in \mathfrak{D}(T_1^*), x_2 \in \mathfrak{D}(T_2) \text{ and } K_1^* x_1 + K_2^* x_2 = 0\}.$$

The inclusion $\mathfrak{D}(\tilde{A}) \subset \mathfrak{D}(A)$ takes place. Indeed, if $x \in \mathfrak{D}(\tilde{A})$ then $x_1 \in \mathfrak{D}(A_1)$, $x_2 \in \mathfrak{D}(A_2)$ and $K_1^* x_1 = 0$, $K_2^* x_2 = 0$. Since $\mathfrak{D}(A_1) \subset \mathfrak{D}(T_1^*)$, $\mathfrak{D}(A_2) \subset \mathfrak{D}(T_2)$ and $K_1^* x_1 + K_2^* x_2 = 0$ we have $\mathfrak{D}(\tilde{A}) \subset \mathfrak{D}(A)$.

From above we can conclude that $\tilde{A} \subset A \subset T$ and $\tilde{A} \subset A \subset T^*$. Moreover, A is a maximal Hermitian part of T and T^* operators. Let A^* be an adjoint to A operator. Then $\mathfrak{D}(A^*) \subset \mathfrak{D}(\tilde{A}^*) = \mathfrak{H}_+$. Let

$$H_+ = \mathfrak{D}(A^*)$$

and construct new rigged space

$$H_+ \subset \mathfrak{H} \subset H_-. \tag{6.12}$$

It is easy to see that the following inclusions take place

$$\begin{array}{c} H_+ \hookrightarrow \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- \hookrightarrow H_- \\ \cap \\ H_- \end{array} \tag{6.13}$$

Let us denote by γ an embedding operator acting from H_+ into $\mathfrak{H}_+$:

$$\gamma : H_+ \hookrightarrow \mathfrak{H}_+, \quad \gamma x = x, \ \forall x \in H_+. \tag{6.14}$$

Let us define an adjoint operator γ^* as $\gamma^* : \mathfrak{H}_- \hookrightarrow H_-$ and operator $\mathbb{A} \in [H_+, H_-]$ as

$$\mathbb{A} = \gamma^* \tilde{\mathbb{A}} \Big|_{H_+}. \tag{6.15}$$

Obviously, $\mathbb{A} \supset T$ and $\mathbb{A}^* \supset T^*$ where $\mathbb{A}^* \in [H_+, H_-]$,

$$\mathbb{A}^* = \gamma^* \tilde{\mathbb{A}}^* \Big|_{H_+}, \tag{6.16}$$

is adjoint to $\mathbb{A}$ operator. The last statement holds since for all $x, y \in H_+$

$$\begin{aligned} (\mathbb{A}x, y) &= (\gamma^* \tilde{\mathbb{A}} x, y) = (\tilde{\mathbb{A}} x, \gamma y) \\ &= (\tilde{\mathbb{A}} x, y) = (x, \tilde{\mathbb{A}}^* y) = (x, \gamma^* \tilde{\mathbb{A}}^* y) \\ &= (x, \mathbb{A}^* y). \end{aligned}$$

Indeed, let $x \in \mathfrak{D}(T)$. Then $\tilde{\mathbb{A}}$ belongs to the $\mathfrak{H}$ space and

$$\mathbb{A}x = \gamma^* \tilde{\mathbb{A}} x = \gamma^* T x = T x \in \mathfrak{H}.$$

Thus, $\mathbb{A} \supset T$. Similarly, $\mathbb{A}^* \supset T^*$. Here we explore the fact that $\gamma^* g = g$ for all $g \in \mathfrak{H}$. Indeed, for all $x \in H_+$, $g \in \mathfrak{H}$

$$(x, g) = (\gamma x, g) = (x, \gamma^* g).$$

Let us show now that operator $\mathbb{A}$ defined by (6.14) can be included in l.s.c.d.s. θ.

$$\begin{aligned} \frac{1}{2i}(\mathbb{A} - \mathbb{A}^*) &= \frac{1}{2i}(\gamma^* \tilde{\mathbb{A}} - \gamma^* \tilde{\mathbb{A}}^*) \\ &= \frac{1}{2i}\gamma^*(\tilde{\mathbb{A}} - \tilde{\mathbb{A}}^*) = \gamma^* \tilde{K} J \tilde{K}^* \\ &= K J K^*, \end{aligned}$$

where $\tilde{K} = K_1 + K_2$, $\tilde{K}^* = K_1^* + K_2^*$ and

$$K = \gamma^* \tilde{K}, \qquad K^* = \tilde{K}^*\Big|_{H_+}. \tag{6.17}$$

For all $e \in E$ and $x \in H_+$ we have

$$\begin{aligned}(Ke, x) &= (\gamma^* \tilde{K} e, x) = (\tilde{K} e, \gamma x) = (\tilde{K} e, x)\\ &= (e, \tilde{x}^*) = (e, K^* x).\end{aligned}$$

and so $K^* = \tilde{K}^*\Big|_{H_+}$.

Thus, $\mathbb{A} \supset T \supset A$, $\mathbb{A}^* \supset T^* \supset A$, $\mathrm{Im}\mathbb{A} = KJK^*$, where K is defined by (6.17) and we can include operator $\mathbb{A}$ in a system

$$\theta = \begin{pmatrix} \mathbb{A} & K & J \\ H_+ \subset \mathfrak{H} \subset H_- & & E \end{pmatrix}. \tag{6.18}$$

Let now

$$\begin{aligned} W_{\theta_1}(z) &= I - 2iK_1^*(\mathbb{A}_1 - zI)^{-1} K_1 J,\\ W_{\theta_2}(z) &= I - 2iK_2^*(\mathbb{A}_2 - zI)^{-1} K_2 J, \end{aligned} \tag{6.19}$$

be transfer operator-valued functions of the systems θ_1 and θ_2 respectively.

We introduce a new auxiliary system

$$\tilde{\theta} = \begin{pmatrix} \tilde{\mathbb{A}} & \tilde{K} & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & E \end{pmatrix} \tag{6.20}$$

where all its components are described above. Let us note that $\tilde{\theta}$ does not exactly satisfy the definition 3.1 of l.s.c.d.s. because operator $\tilde{A}$ is not the maximal Hermitian part of T and T^* and thus $T \notin \Lambda_{\tilde{A}}$. System $\tilde{\theta}$ will, however, suffice our purposes.

Let

$$W_{\tilde{\theta}}(z) = I - 2i\tilde{K}^*(\tilde{\mathbb{A}} - zI)^{-1} \tilde{K} J, \tag{6.21}$$

be the transfer operator-valued function of the system $\tilde{\theta}$. According to the first equality (6.3) we have

$$\tilde{\mathbb{A}} - zI = (\mathbb{A}_1 - zI)P_1^+ + (\mathbb{A}_2 - zI)P_2^+ + 2iK_1 J K_2^+ P_2^+.$$

Direct check shows that

$$\begin{aligned}(\tilde{\mathbb{A}} - zI)^{-1} =&(\mathbb{A}_1 - zI)^{-1} P_1^- + (\mathbb{A}_2 - zI)^{-1} P_2^-\\ &- 2i(\mathbb{A}_1 - zI)^{-1} K_1 J K_2 (\mathbb{A}_2 - zI)^{-1} P_2^-.\end{aligned} \tag{6.22}$$

This implies

$$\begin{aligned} W_{\tilde{\theta}}(z) =&I - 2iK^*(\tilde{\mathbb{A}} - zI)^{-1} KJ\\ =&I - 2iK^* P_1^-(\mathbb{A}_1 - zI)^{-1} P_1^- KJ - 2iK^* P_2^-(\mathbb{A}_2 - zI)^{-1} P_2^- KJ\\ &+ (2i)^2 K^* P_1^-(\mathbb{A}_1 - zI)^{-1} K_1 J K_2^*(\mathbb{A}_2 - zI)^{-1} P_2^- KJ\\ =&I + 2iK_1^*(\mathbb{A}_1 - zI)^{-1} K_1 J - 2iK_2^*(\mathbb{A}_2 - zI)^{-1} K_2 J\\ &+ (2i)^2 K_1^*(\mathbb{A}_1 - zI)^{-1} K_1 J K_2^*(\mathbb{A}_2 - zI)^{-1} K_2 J\\ =&(I - 2iK_1^*(\mathbb{A}_1 - zI)^{-1} K_1 J)(I - 2iK_2^*(\mathbb{A}_2 - zI)^{-1} K_2 J)\\ =&W_{\theta_1}(z) \cdot W_{\theta_2}(z). \end{aligned}$$

In other words we have just shown that

$$W_{\tilde{\theta}}(z) = W_{\theta_1}(z) \cdot W_{\theta_2}(z). \tag{6.23}$$

Let now

$$W_\theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ, \tag{6.24}$$

be a transfer operator-valued function of the system θ defined by (6.18). We will show now that

$$W_\theta(z) = W_{\tilde{\theta}}(z) = W_{\theta_1}(z) \cdot W_{\theta_2}(z). \tag{6.25}$$

It is enough to show that

$$K^*(\mathbb{A} - zI)^{-1}K = \tilde{K}^*(\tilde{\mathbb{A}} - zI)^{-1}\tilde{K}, \tag{6.26}$$

and the rest will follow from above. Consider the difference

$$\begin{aligned} K^*(\mathbb{A} - zI)^{-1}K &- \tilde{K}^*(\tilde{\mathbb{A}} - zI)^{-1}\tilde{K} \\ &= \tilde{K}^*(\gamma^*\tilde{\mathbb{A}} - zI)^{-1}\gamma^*\tilde{K} - \tilde{K}^*(\tilde{\mathbb{A}} - zI)^{-1}\tilde{K} \\ &= \tilde{K}^*[(\gamma^*\tilde{\mathbb{A}} - zI)^{-1}\gamma^* - (\tilde{\mathbb{A}} - zI)^{-1}]\tilde{K}. \end{aligned}$$

Let $e \in E$. We choose a sequence of elements g_n from $\mathfrak{H}$ such that $g_n \to \tilde{K}e$. Then

$$\tilde{K}^*[(\gamma^*\tilde{\mathbb{A}} - zI)^{-1}\gamma^* - (\tilde{\mathbb{A}} - zI)^{-1}]g_n = 0,$$

since $\gamma^* g_n = g_n$ and T is a common quasi-kernel of operators $\tilde{\mathbb{A}}$ and $\mathbb{A}$. Now let $g_n \to \tilde{K}e$ as $n \to \infty$ we have as a limit case

$$\tilde{K}^*\big((\gamma^*\tilde{\mathbb{A}} - zI)^{-1}\gamma^* - (\tilde{\mathbb{A}} - zI)^{-1}\big)\tilde{K}e = 0. \tag{6.27}$$

Since $\mathfrak{H}$ is dense in $\mathfrak{H}_-$ we can repeat that procedure for any $e \in E$. Hence

$$K^*(\mathbb{A} - zI)^{-1}K = \tilde{K}^*(\tilde{\mathbb{A}} - zI)^{-1}\tilde{K}$$

and (6.25) yields

$$W_\theta(z) = W_{\theta_1}(z) \cdot W_{\theta_2}(z).$$

Let us show now that $\mathbb{A}$ is a correct $(*)$-extension of operator T. In order to prove we show that real part of $\mathbb{A}$ has a self-adjoint quasi-kernel. Let

$$\mathbb{A}_{1R} = \frac{\mathbb{A}_1 + \mathbb{A}_1^*}{2} \quad \text{and} \quad \mathbb{A}_{2R} = \frac{\mathbb{A}_2 + \mathbb{A}_2^*}{2},$$

be the real parts of operators $\mathbb{A}_1$ and $\mathbb{A}_2$ respectively. From (6.3) we obtain

$$\tilde{\mathbb{A}}_R = \mathbb{A}_{1R}P_1^+ + \mathbb{A}_{2R}P_2^+ + 2iK_1JK_2^*P_2^+. \tag{6.28}$$

Since both $\mathbb{A}_{1R}$ and $\mathbb{A}_{2R}$ are strong self-adjoint bi-extensions of operator A_1 and A_2 then $(\mathbb{A}_{1R} - zI)^{-1}$ and $(\mathbb{A}_{2R} - zI)^{-1}$ are $(-,\cdot)$ and hence $(-,-)$-continuous (see Theorem 2.5). We note that operator $\tilde{A} = A_1 \oplus A_2$ has finite and equal deficiency indices. If we use (6.28) and write formula (6.22) for the operator $\tilde{\mathbb{A}}_R$ we get

$$\begin{aligned} (\tilde{\mathbb{A}}_R - zI)^{-1} =& (\mathbb{A}_{1R} - zI)^{-1}P_1^- + (\mathbb{A}_{2R} - zI)^{-1}P_2^- \\ & - 2i(\mathbb{A}_{1R} - zI)^{-1}K_1JK_2(\mathbb{A}_{2R} - zI)^{-1}P_2^-. \end{aligned}$$

Next according to the Corollary 2.6 we conclude that $\tilde{\mathbb{A}}_R$ is a strong self-adjoint bi-extension of the operator $\tilde{A}$. Therefore, $\tilde{\mathbb{A}}_R$ has a self-adjoint quasi-kernel $B = B^*$ and the following inclusion takes place

$$\tilde{\mathbb{A}}_R \supset B = B^* \supset \tilde{A}.$$

Our objective is to show that

$$\mathbb{A}_R \supset B = B^* \supset A. \tag{6.29}$$

Let us note that $\mathfrak{D}(A^*) = H_+ \subset \mathfrak{H}_+ = \mathfrak{D}(\tilde{A}^*)$. Hence to prove (6.29) it is sufficient to show that

$$\tilde{\mathbb{A}}_R \supset B = B^* \supset A \supset \tilde{A}.$$

First we show that $\mathfrak{D}(B) \supset \mathfrak{D}(A)$. Let assume that

$$\mathfrak{D}(B) \cap \mathfrak{D}(A) \neq \mathfrak{D}(A),$$

which means that there exists an element $x_0 \in \mathfrak{D}(A)$ such that $x_0 \notin \mathfrak{D}(B)$. Suppose

$$F = \{\lambda x_0,\ \lambda \in \mathbb{C}\},$$

and $\mathfrak{D}(\tilde{B}) = \mathfrak{D}(B) + F$. We define new operator $\tilde{B}$ on $\mathfrak{D}(\tilde{B})$ by the formula

$$\tilde{B}x = Bx_B + Ax_F,$$

where $x \in \mathfrak{D}(\tilde{B})$, $x = x_B + x_F$, $x_B \in \mathfrak{D}(B)$, $x_F \in F$. Then the form $(\tilde{B}x, x) = (\tilde{\mathbb{A}}_R x, x)$ is real for all $x \in \mathfrak{D}(\tilde{B})$. This means that self-adjoint operator B admits a symmetric extension $\tilde{B}$ which is impossible. Hence, we get a contradiction and $\mathfrak{D}(B) \supset \mathfrak{D}(A)$. This would imply $\mathfrak{D}(B) = \mathfrak{D}(B^*) \subset \mathfrak{D}(A^*) = H_+$ or $B \subset \mathbb{A}_R$. Putting all this together we conclude that formula (6.29) holds and $\mathbb{A}_R$ is indeed a strong self-adjoint bi-extension of the operator A. We can conclude now that system θ defined by (6.18) is l.s.c.d.s.

Definition 6.2 The system

$$\theta = \begin{pmatrix} \mathbb{A} & K & J \\ H_+ \subset \mathfrak{H} \subset H_- & & E \end{pmatrix} \tag{6.30}$$

is called a product of two systems

$$\theta_1 = \begin{pmatrix} \mathbb{A}_1 & K_1 & J_1 \\ \mathfrak{H}_{+1} \subset \mathfrak{H}_1 \subset \mathfrak{H}_{-1} & & E_1 \end{pmatrix} \quad \text{and} \quad \theta_2 = \begin{pmatrix} \mathbb{A}_2 & K_2 & J_2 \\ \mathfrak{H}_{+2} \subset \mathfrak{H}_2 \subset \mathfrak{H}_{-2} & & E_2 \end{pmatrix} \tag{6.31}$$

if operators $\mathbb{A}$, K and rigged space $H_+ \subset \mathfrak{H} \subset H_-$ are defined by the formulas (6.15), (6.17) and (6.12), respectively.

Theorem 6.3 *Let system θ be the product of two systems θ_1 and θ_2. Then if λ is a regular point for operators $\mathbb{A}_1$ and $\mathbb{A}_2$ then*

$$W_\theta(\lambda) = W_{\theta_1}(\lambda) \cdot W_{\theta_2}(\lambda).$$

The proof of the Theorem 6.3 was constructively obtained above.

Theorem 6.4 *Let operator-valued functions $W_1(z)$ and $W_2(z)$ belong to the class $\Omega(R, J)$. Then operator-valued function*

$$W(z) = W_1(z) \cdot W_2(z)$$

also belongs to the class $\Omega(R, J)$.

Proof Since operator-valued functions $W_1(z)$ and $W_2(z)$ belong to the class $\Omega(R, J)$ then there exist two l.s.c.d.s. θ_1 and θ_2 of the form (6.27) such that $W_1(z) = W_{\theta_1}(z)$ and $W_2(z) = W_{\theta_2}(z)$ in some neighborhood of the point $(-i)$. Let l.s.c.d.s. θ be the product of θ_1 and θ_2. Then according to the Theorem 6.3

$$W_\theta(z) = W_{\theta_1}(z) \cdot W_{\theta_2}(z) = W_1(z) \cdot W_2(z) = W(z).$$

This means that for the functions $W(z)$ there exists a system θ such that $W(z) = W_\theta(z)$ in some neighborhood of $(-i)$. Therefore, $W(z) = W_1(z) \cdot W_2(z)$ belongs to $\Omega(R, J)$. □

The next theorem establishes similar result for the class $\Omega_0(R, J)$.

Theorem 6.5 *Let operator-valued functions $W_1(z)$ and $W_2(z)$ belong to the class $\Omega_0(R, J)$. Then operator-valued function*

$$W(z) = W_1(z) \cdot W_2(z)$$

is also a member of the class $\Omega_0(R, J)$. [2]

Proof According to the Theorem 6.4 we have $W(z)$ belongs to $\Omega(R, J)$. Therefore, there exists a system

$$\theta = \begin{pmatrix} \mathbb{A} & K & J \\ H_+ \subset \mathfrak{H} \subset H_- & & E \end{pmatrix} \tag{6.32}$$

such that $W(z) = W_\theta(z)$ in some neighborhood of $(-i)$. So, it would be enough to show that if $\mathbb{A} \supset T \supset A$, $\mathbb{A}^* \supset T^* \supset A$ are correspondent elements of θ then $\overline{\mathfrak{D}(A)} = \mathfrak{H}$ and $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$.

Since $W_1(z)$ and $W_2(z)$ both belong to the class $\Omega_0(R, J)$ then correspondent systems θ_1 and θ_2 have a property that $\overline{\mathfrak{D}(A_1)} = \mathfrak{H}_1$ and $\overline{\mathfrak{D}(A_2)} = \mathfrak{H}_2$. Let operator $\tilde{A}$ be defined by (4.3) and

$$\mathfrak{D}(\tilde{A}) = \mathfrak{D}(A_1) \oplus \mathfrak{D}(A_2). \tag{6.33}$$

Considering the closure of the equality (6.33) yields $\overline{\mathfrak{D}(\tilde{A})} = \mathfrak{H}_1 \oplus \mathfrak{H}_2 = \mathfrak{H}$. As it was shown above $\tilde{A} \subset A$ or $\mathfrak{D}(\tilde{A}) \subset \mathfrak{D}(A) \subset \mathfrak{H}$. Hence, $\overline{\mathfrak{D}(A)} = \mathfrak{H}$.

It was shown in the proof of Theorem 4.8 (see [**8**]) that $\overline{\mathfrak{D}(A)} = \mathfrak{H}$ already implies that $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$. Thus, $W_\theta(z) = W_{\theta_1}(z) \cdot W_{\theta_2}(z)$ belongs to the class $\Omega_0(R, J)$. □

Theorem 6.6 *Let operator-valued functions $W_1(z)$ and $W_2(z)$ belong to the class $\Omega_1(R, J)$. Then their product*

$$W(z) = W_1(z) \cdot W_2(z)$$

also belongs to the class $\Omega_1(R, J)$.

Proof Using the same argument as in the theorem above we conclude that $W(z)$ is a member of $\Omega(R, J)$ class and is realizable by the system θ of the type (6.32) such that $W(z) = W_\theta(z)$ in a neighborhood of the point $(-i)$. What remain is to show that system θ has a property $\overline{\mathfrak{D}(A)} \neq \mathfrak{H}$ and $\mathfrak{D}(T) = \mathfrak{D}(T^*)$.

Since both $W_1(z)$ and $W_2(z)$ belong to the class $\Omega_1(R, J)$ then correspondent systems θ_1 and θ_2 posses properties $\overline{\mathfrak{D}(A_1)} \neq \mathfrak{H}$, $\mathfrak{D}(T_1) = \mathfrak{D}(T_1^*)$ and $\overline{\mathfrak{D}(A_2)} \neq \mathfrak{H}$, $\mathfrak{D}(T_2) = \mathfrak{D}(T_2^*)$. Due to the Theorem 6.4 $\theta = \theta_1 \cdot \theta_2$. In the proof of the Theorem 4.9 (see [**8**]) we have shown that systems with above condition have reduced form. Namely,

$$\theta_1 = \begin{pmatrix} T_1 & K_1 & J_1 \\ H_{+1} \subset H_1 \subset H_{-1} & & E_1 \end{pmatrix} \quad \text{and} \quad \theta_2 = \begin{pmatrix} T_2 & K_2 & J_2 \\ H_{+2} \subset H_2 \subset H_{-2} & & E_2 \end{pmatrix}$$

[2]Based on a theorem by Yu.M. Arlinskiĭ and one of the autors (see [**2**]), it can be shown that under the conditions of the theorem 6.5 not only $W_1(z)W_2(z)$ but also $UW_1(z)W_2(z)V$ belongs to the class $\Omega_0(R, J)$. Here U and V are arbitrary J-unitary operators acting on E.

where

$$\mathrm{Im}T_m = \frac{T_m - T_m^*}{2i} = K_m J K_m^*, \qquad (m = 1, 2).$$

Consequently, the main operator of system $\theta = \theta_1 \cdot \theta_2$ is determined by formulas

$$\begin{aligned} T &= T_1 P_1 + T_2 P_2 + 2iK_1 J K_2^* P_2, \\ T^* &= T_1^* P_1 + T_2^* P_2 + 2iK_1 J K_2^* P_2, \end{aligned}$$

and $K = K_1 + K_2$. Using the fact that $\mathfrak{D}(T_1) = \mathfrak{D}(T_1^*)$ and $\mathfrak{D}(T_2) = \mathfrak{D}(T_2^*)$ we conclude that $\mathfrak{D}(T) = \mathfrak{D}(T^*)$. This implies (see [**8**]) that $\overline{\mathfrak{D}(A)} \neq \mathfrak{H}$. Therefore, $W(z)$ belongs to the class $\Omega_1(R, J)$. $\square$

The following correspondent result for the class $\Omega_{01}(R, J)$ is not that straightforward and additional condition is required.

Theorem 6.7 *Let operator-valued functions $W_1(z)$ and $W_2(z)$ belong to the class $\Omega_{01}(R, J)$. Then their product*

$$W(z) = W_1(z) \cdot W_2(z)$$

belongs to the class $\Omega_{01}(R, J)$ if and only if the set

$$\mathfrak{D} = \left\{ x = x_1 + x_2 \in \mathfrak{H}_1 \oplus \mathfrak{H}_2 \middle| \, x_1 \in \mathfrak{D}(T_1^*), x_2 \in \mathfrak{D}(T_2), K_1^* x_1 + K_2^* x_2 = 0 \right\}, \tag{6.34}$$

is not dense in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Here T_1, T_2, K_1, K_2, $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are correspondent elements of the systems θ_1 and θ_2 related to the functions $W_1(z)$ and $W_2(z)$.

Proof Since $\Omega(R, J)$ is a union of three distinct classes $\Omega_0(R, J)$, $\Omega_1(R, J)$ and $\Omega_{01}(R, J)$, Theorem 6.4 guarantees that $W(z)$ belongs to one of the indicated subclasses. The set $\mathfrak{D}$ defined by (6.34) actually coincides with the domain of operator A defined in (6.8). Therefore, since $\mathfrak{D}$ is not dense in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ then $\overline{\mathfrak{D}(A)} \neq \mathfrak{H}$ and $W(z)$ is certainly not in the $\Omega_0(R, J)$ class.

Let us assume that $W(z)$ belongs to $\Omega_1(R, J)$. Then the system $\theta = \theta_1 \cdot \theta_2$ has a property $\mathfrak{H}_+ = \mathfrak{D}(T) = \mathfrak{D}(T^*)$. The operator T here is actually a quasi-kernel of the main operator $\mathbb{A}$ of the system θ. That means that for all $x \in \mathfrak{H}_+$, $x = x_1 + x_2$, $x_1 \in \mathfrak{H}_{+1}$, $x_2 \in \mathfrak{H}_{+2}$

$$\begin{aligned} \mathbb{A}_1 x_1 + \mathbb{A}_2 x_2 - 2iK_1 J K_2^* x_2 \in \mathfrak{H}, \\ \mathbb{A}_2^* x_2 + \mathbb{A}_2^* x_2 - 2iK_2 J K_1^* x_2 \in \mathfrak{H}, \end{aligned} \tag{6.35}$$

where $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and all operators belong to the correspondent systems θ_1 and θ_2. Since x_2 is an arbitrary element of $\mathfrak{H}_{+2}$ then we can choose it equal to 0. Then first relation yields $x_1 \in \mathfrak{D}(T_1)$ for all $x_1 \in \mathfrak{H}_{+1}$. Because x_1 is arbitrary we have that

$$\mathfrak{D}(T_1) = \mathfrak{H}_{+1} = \mathfrak{D}(A_1^*).$$

Considering the fact that $W_1(z)$ is a member of $\Omega_{01}(R, J)$ class we get a contradiction. Hence, the product of $W_1(z)$ and $W_2(z)$ under the assumption of the theorem belongs to the class $\Omega_{01}(R, J)$. $\square$

Remark 6.8 It is not hard to show that if the set $\mathfrak{D}$ in the statement of the Theorem 6.7 is dense in $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ then $W(z) = W_1(z) \cdot W_2(z)$ belongs to the class $\Omega_0(R, J)$.

The theorem below describes properties of the mixed products of two operator-valued functions of $\Omega(R, J)$ class.

Theorem 6.9 *Let operator-valued functions $W_1(z)$ and $W_2(z)$ belong to the classes $\Omega_0(R,J)$ and $\Omega_1(R,J)$, respectively. Then their product*

$$W(z)=W_1(z)\cdot W_2(z)$$

belongs to the class $\Omega_{01}(R,J)$ if and only if the set $\mathfrak{D}$ of the form (6.34) *is not dense in $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$.*

We omit the proof of this theorem because it is similar to the one of the Theorem 6.7. As before we should note that if the set $\mathfrak{D}$ in the statement of the Theorem 6.9 is not dense in $\mathfrak{H}_1\oplus\mathfrak{H}_2$ then $W(z)=W_1(z)\cdot W_2(z)$ belongs to the class $\Omega_0(R,J)$. Furthermore, Theorem 6.9 holds even if we consider product $W(z)=W_2(z)\cdot W_1(z)$.

Theorem 6.10 *Let operator-valued functions $W_1(z)$ and $W_2(z)$ belong to the classes $\Omega_0(R,J)$ and $\Omega_1(R,J)$, respectively. Let also*

$$W_1(z)\cdot W_2(z)=W_2(z)\cdot W_1(z)=W(z). \tag{6.36}$$

Then operator-valued function $W(z)$ belongs to the class $\Omega_{01}(R,J)$.

Proof Condition (6.36) implies that

$$\tilde{\mathbb{A}}=\mathbb{A}_1P_1^+ + \mathbb{A}_2P_2^+ + 2iK_1JK_2^*P_2^+ = \mathbb{A}_2P_2^+ + \mathbb{A}_1P_1^+ + 2iK_2JK_1^*P_1^+.$$

Canceling yields

$$K_1JK_2^*P_2^+ = K_2JK_1^*P_1^+.$$

Left and right hand sides of this equality belong to H_{-1} and H_{-2}, respectively. Hence the equality may hold only if

$$K_1JK_2^*P_2^+ = K_2JK_1^*P_1^+ = 0.$$

Thus,

$$\tilde{\mathbb{A}}=\mathbb{A}_1P_1^+ + \mathbb{A}_2P_2^+, \tag{6.37}$$

and we are actually dealing with operator $\tilde{\mathbb{A}}$ of block-diagonal structure.

Now let $x_T=x_1+x_2$ be an element of $\mathfrak{D}(T)$, then

$$Tx_T=\tilde{\mathbb{A}}x_T=\mathbb{A}_1x_1+\mathbb{A}_2x_2\in\mathfrak{H},$$

but $\mathbb{A}_2x_2\in\mathfrak{H}$, and therefore, $\mathbb{A}_1x_1\in\mathfrak{H}$, or

$$Tx_T=T_1x_1+T_2x_2, \tag{6.38}$$

Similarly,

$$T^*x_{T^*}=T_1^*x_1+T_2^*x_2, \tag{6.39}$$

where $x_{T^*}=x_1+x_2$ is an element of $\mathfrak{D}(T^*)$. In other words we have just shown that $x_T\in\mathfrak{D}(T)$ implies $x_1\in\mathfrak{D}(T_1)$ and $x_2\in\mathfrak{D}(T_2)$, where $x_T=x_1+x_2$. Conversely, if $x_1\in\mathfrak{D}(T_1)$ and $x_2\in\mathfrak{D}(T_2)$, then $x_T=x_1+x_2\in\mathfrak{D}(T)$, i.e.

$$\mathfrak{D}(T)=\mathfrak{D}(T_1)\oplus\mathfrak{D}(T_2). \tag{6.40}$$

Similarly shown,

$$\mathfrak{D}(T^*)=\mathfrak{D}(T_1^*)\oplus\mathfrak{D}(T_2^*). \tag{6.41}$$

But since $\mathfrak{D}(T_1)\neq\mathfrak{D}(T_1^*)$ and $\mathfrak{D}(T_2)=\mathfrak{D}(T_2^*)$ then $\mathfrak{D}(T)\neq\mathfrak{D}(T^*)$.

It is also not hard to see that under this circumstances

$$\mathfrak{D}(A)=\mathfrak{D}(A_1)\oplus\mathfrak{D}(A_2). \tag{6.42}$$

Hence, if $\overline{\mathfrak{D}(A_2)}\neq\mathfrak{H}_2$ then $\overline{\mathfrak{D}(A)}\neq\mathfrak{H}$. It follows then that $W(z)$ belongs to the class $\Omega_{01}(R,J)$. $\square$

7 Example

In this section we present an example of two l.s.c.d. systems with the transfer functions that fall into classes $\Omega_1(R,J)$ and $\Omega_0(R,J)$. Then we will show that the product of these two transfer functions belongs to the class $\Omega_0(R,J)$.

Let us consider an operator

$$Ty = -y'', \tag{7.1}$$

defined on the set

$$\mathfrak{D}(T) = \{y(t) \in L^2_{[0,l]} : y''(t) \in L^2_{[0,l]}, y(0) = y'(0) = 0\}.$$

It is easy to show that its adjoint operator

$$T^*y = -y'', \tag{7.2}$$

is defined on the set

$$\mathfrak{D}(T^*) = \{y(t) \in L^2_{[0,l]} : y''(t) \in L^2_{[0,l]}, y(l) = y'(l) = 0\}.$$

Solution of the initial value problem provides us with the inverse operator

$$T^{-1}f = \int_0^x (t-x)f(t)dt, \quad f \in L^2_{[0,l]}. \tag{7.3}$$

Let $\xi(x) = \|\varphi_1(x) \quad \varphi_2(x)\|$ be a row vector whose entries are

$$\varphi_1(x) = \frac{1}{\sqrt{2}} \quad \text{and} \quad \varphi_2(x) = \frac{x}{\sqrt{2}}, \tag{7.4}$$

and let

$$J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Obviously, $J = J^* = J^{-1}$. Then (7.3) can be re-written (see [**12**]) as

$$T^{-1} = -2i \int_0^x f(t)\xi(t)\, dt J\xi^*(x). \tag{7.5}$$

Similarly,

$$(T^{-1})^*f = 2i \int_x^l f(t)\xi(t)\, dt J\xi^*(x).$$

Now we can find

$$\frac{T^{-1} - (T^{-1})^*}{2i} f(x) = \int_0^l f(t)\xi(t)\, dt J\xi^*(x) = \sum_{\alpha,\beta=1}^{2} (f,\varphi_\alpha) j_{\alpha\beta}\varphi_\beta(x).$$

Here $j_{\alpha\beta}$ is an element of J and $\varphi_i(x)$, $i = 1,2$ are defined above.

According to [**11**] T^{-1} can be included into the system

$$\theta_1 = \begin{pmatrix} T^{-1} & K_1 & J \\ L^2_{[0,l]} & & \mathbb{C}^2 \end{pmatrix}, \tag{7.6}$$

where K_1 is a channel operator that is going to be described below. First, we find an operator K_1^* for the system θ_1. Let us remind that

$$\frac{T^{-1} - (T^{-1})^*}{2i} f = \sum_{\alpha,\beta=1}^{2} (f,\varphi_\alpha) j_{\alpha\beta}\varphi_\beta = K_1 J K_1^*. \tag{7.7}$$

Using this relation we can see that

$$K_1^* f = \begin{pmatrix} (f,\varphi_1) \\ \\ (f,\varphi_2) \end{pmatrix}, \tag{7.8}$$

where

$$(f,\varphi_1) = \frac{1}{\sqrt{2}} \int_0^l f(t)dt \quad \text{and} \quad (f,\varphi_2) = \frac{1}{\sqrt{2}} \int_0^l t f(t)dt. \tag{7.9}$$

The operator $K_1 : \mathbb{C}^2 \to L^2_{[0,l]}$ then

$$K_1 \begin{pmatrix} c_1 \\ \\ c_2 \end{pmatrix} = c_1\varphi_1(x) + c_2\varphi_2(x). \tag{7.10}$$

Let $W_{\theta_1}(\lambda)$ be a characteristic operator-valued function of the system θ_1. Then [**12**] it is represented by the formula

$$W_{\theta_1}(\lambda) = I - 2i\|(T^{-1} - \lambda I)^{-1}\varphi_\alpha, \varphi_\beta\|J, \quad \alpha,\beta = 1,2, \tag{7.11}$$

or

$$W_{\theta_1}(\lambda) = I - 2i \int_0^l \left[(T^{-1} - \lambda I)^{-1}\xi^*(x)\right]\xi(x)\,dx J. \tag{7.12}$$

Hence, in order to write down $W_{\theta_1}(\lambda)$ we have to find the resolvent $(T^{-1} - \lambda I)^{-1}$ values on φ_i, $(i = 1,2)$. Using (7.4) we are solving

$$(T^{-1} - \lambda I)^{-1} f(x) = \frac{1}{\sqrt{2}}, \tag{7.13}$$

and

$$(T^{-1} - \lambda I)^{-1} f(x) = \frac{x}{\sqrt{2}}, \tag{7.14}$$

for $f(x)$. In this case (7.13) yields

$$(T^{-1} - \lambda I)^{-1}\varphi_1(x) = -\frac{1}{2\sqrt{2}\lambda} e^{\frac{i}{\sqrt{\lambda}}x} - \frac{1}{2\sqrt{2}\lambda} e^{-\frac{i}{\sqrt{\lambda}}x},$$

and (7.14) yields

$$(T^{-1} - \lambda I)^{-1}\varphi_2(x) = -\frac{1}{2\sqrt{2}i\sqrt{\lambda}} e^{\frac{i}{\sqrt{\lambda}}x} - \frac{1}{2\sqrt{2}i\sqrt{\lambda}} e^{-\frac{i}{\sqrt{\lambda}}x}.$$

After the routine calculations we end up with the matrix

$$\|(T^{-1} - \lambda I)^{-1}\varphi_\alpha, \varphi_\beta\| = \begin{pmatrix} \frac{1}{i\sqrt{\lambda}}\Gamma(\lambda) & \left(\frac{l}{i\sqrt{\lambda}} - 1\right)\Gamma(\lambda) \\ \\ \Gamma(\lambda) & \left(l - i\sqrt{\lambda}\right)\Gamma(\lambda) \end{pmatrix}, \tag{7.15}$$

where

$$\Gamma(\lambda) = \frac{e^{\frac{i}{\sqrt{\lambda}}l} - e^{-\frac{i}{\sqrt{\lambda}}l}}{2\sqrt{2}}. \tag{7.16}$$

Then

$$\begin{aligned} I - 2i\|(T^{-1} - \lambda I)^{-1}\varphi_\alpha, \varphi_\beta\|J \\ &= \begin{pmatrix} 1 - \left(1 - \frac{l}{i\sqrt{\lambda}}\right)\Gamma(\lambda) & -\frac{1}{i\sqrt{\lambda}}\Gamma(\lambda) \\ \\ (l - i\sqrt{\lambda})\Gamma(\lambda) & 1 - \Gamma(\lambda) \end{pmatrix} \\ &= W_{\theta_1}(\lambda), \end{aligned}$$

and

$$W_{\theta_1}\left(\frac{1}{\lambda}\right) = \begin{pmatrix} 1 - (1 - il\sqrt{\lambda})\Gamma\left(\frac{1}{\lambda}\right) & -i\sqrt{\lambda}\Gamma\left(\frac{1}{\lambda}\right) \\ \\ (l - \frac{i}{\sqrt{\lambda}}\Gamma\left(\frac{1}{\lambda}\right) & 1 - \Gamma\left(\frac{1}{\lambda}\right) \end{pmatrix}. \tag{7.17}$$

Thus, we just have found a characteristic operator-valued function of the system θ_1. Let us note that $\mathfrak{D}(T^{-1}) = \mathfrak{D}((T^{-1})^*)$ and therefore $W_{\theta_1}(\lambda)$ can be related to the class $\Omega_1(R, J)$.

According to [**31**] there exists a bi-extension $\mathbb{A}$ of the operator T and the system

$$\theta_2 = \begin{pmatrix} \mathbb{A} & K_2 & J \\ \mathfrak{H}_+ \subset L^2_{[0,l]} \subset \mathfrak{H}_- & & \mathbb{C}^2 \end{pmatrix}, \tag{7.18}$$

such that

$$W_{\theta_2}(\lambda) = W_{\theta_1}\left(\frac{1}{\lambda}\right).$$

Here $A_2 = -y''$ with

$$\mathfrak{D}(A_2) = \left\{y \in L^2_{[0,l]} : y'' \in L^2_{[0,l]}, y(0) = y(l) = y'(0) = y'(l) = 0\right\}. \tag{7.19}$$

Then $A_2 \subset T \subset \mathbb{A}$, $A_2 \subset T^* \subset \mathbb{A}^*$, $\overline{\mathfrak{D}(A_2)} = L^2_{[0,l]}$, $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$. Furthermore, the transfer operator-valued function

$$W_{\theta_2}(\lambda) = W_{\theta_1}\left(\frac{1}{\lambda}\right), \tag{7.20}$$

belongs to the class $\Omega_0(R, J)$.

We use similar approach on the system θ_2. In order to find generalized vectors $\hat{\varphi}_i \in \mathfrak{H}_-$ $(i = 1, 2)$ we explore [**31**],[**32**] the relations $(A_2^* y, \varphi_1) = (y, \hat{\varphi}_1)$ and $(A_2^* y, \varphi_2) = (y, \hat{\varphi}_2)$, $y \in \mathfrak{H}_+$. It turns out that

$$\hat{\varphi}_1 = \frac{1}{\sqrt{2}}[\delta'(x - l) - \delta'(x)]$$

and

$$\hat{\varphi}_2 = \frac{1}{\sqrt{2}}[l\delta'(x - l) - \delta(x - l) - \delta(x)],$$

where $\delta(x)$ is the delta-function. Then

$$\frac{\mathbb{A} - \mathbb{A}^*}{2i} = \sum_{\alpha,\beta=1}^{2} (\cdot, \hat{\varphi}_\alpha) j_{\alpha\beta} \hat{\varphi}_\beta. \tag{7.21}$$

This implies

$$K_2^* g = \begin{pmatrix} (g, \hat{\varphi}_1) \\ \\ (g, \hat{\varphi}_2) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} g'(0) - g'(l) \\ \\ g(l) - g(0) - lg'(l) \end{pmatrix}. \tag{7.22}$$

Now let $W(\lambda)=W_{\theta_1}(\lambda)\cdot W_{\theta_2}(\lambda)$. We will show that $W(\lambda)$ belongs to $\Omega_0(R,J)$. In order to do that according to the Theorem 6.7 we must show that

$$\mathfrak{D}=\left\{x_1\in\mathfrak{D}((T^{-1})^*),x_2\in\mathfrak{D}(T),K_1^*x_1+K_2^*x_2=0\right\},$$

is dense in $L^2_{[0,l]}\otimes L^2_{[0,l]}$.

$$K_1^*x_1+K_2^*x_2=0,$$

implies that

$$\begin{aligned}\int_0^l x_1(t)dt+x_2'(0)-x_2'(l)=0,\\ \int_0^l tx_1(t)dt+x_2(l)-x_2(0)-lx_2'(l)=0.\end{aligned}\tag{7.23}$$

Taking into account the fact that $x_1(t)\in\mathfrak{D}((T^{-1})^*)$ and $x_2(t)\in\mathfrak{D}(T)$ we simplify (7.23) and get

$$\begin{aligned}\int_0^l x_1(t)dt-x_2'(l)=0,\\ \int_0^l tx_1(t)dt+x_2(l)-lx_2'(l)=0.\end{aligned}\tag{7.24}$$

We shall show that the set of vector-functions $\begin{pmatrix}x_1(t) & x_2(t)\end{pmatrix}$ with condition (7.24) is dense in $L^2_{[0,l]}\otimes L^2_{[0,l]}$. Let $\begin{pmatrix}y_1(t) & y_2(t)\end{pmatrix}$ be an arbitrary vector-function from $L^2_{[0,l]}\otimes L^2_{[0,l]}$. It is known [**25**] that the set of differential functions with fixed value of function and its derivative at the point is dense in $L^2_{[0,l]}$. Then there exists a sequence $x_2^{(n)}(t)\in L^2_{[0,l]}$ such that

$$\begin{aligned}{x'}_2^{(n)}(l)=\int_0^l y_1(t)dt,\quad\forall n\in\mathbb{N},\\ l{x'}_2^{(n)}(l)-x_2^{(n)}(l)=\int_0^l ty_1(t)dt,\quad\forall n\in\mathbb{N},\end{aligned}$$

and $\|y_2(t)-x_2^{(n)}(t)\|\to 0$ when $n\to\infty$. Thus an arbitrary vector-function $\begin{pmatrix}y_1(t) & y_2(t)\end{pmatrix}$ was approximated by the sequence of elements form $\mathfrak{D}$. Hence $\mathfrak{D}$ is dense in $L^2_{[0,l]}\otimes L^2_{[0,l]}$ and operator-valued function $W(\lambda)=W_{\theta_1}(\lambda)\cdot W_{\theta_2}(\lambda)$ belongs to the class $\Omega_0(R,J)$.

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