ON CLASSES OF REALIZABLE OPERATOR-VALUED R-FUNCTIONS

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In this paper we consider realization problems (see [5]–[7]) for operator-valued R-functions acting on a Hilbert space E ($\dim E < \infty$) as linear-fractional transformations of the transfer operator-valued functions (characteristic functions) of linear stationary conservative dynamic systems (Brodskiĭ-Livšic rigged operator colligations). We specialize three subclasses of the class of all realizable operator-valued R-functions [7]. We give complete proofs of direct and inverse realization theorems for each subclass announced in [5], [6].

1. Introduction

Realization theory of different classes of operator-valued (matrix-valued) functions as transfer operator-functions of linear systems plays an important role in modern operator and systems theory. Almost all realizations in the modern theory of non-selfadjoint operators and its applications deal with systems (operator colligations) in which the main operators are *bounded* linear operators [8], [10-16], [17], [23]. The realization with an *unbounded* operator as a main operator in a corresponding system has not been investigated thoroughly because of a number of essential difficulties usually related to unbounded non-selfadjoint operators.

This paper is the logical continuation of the results stated and proved in [7]. We consider realization problems for operator-valued R-functions acting on a finite dimensional Hilbert space E as linear-fractional transformations of the transfer operator-functions of linear stationary conservative dynamic systems (l.s.c.d.s.) θ of the form

$$\begin{cases} (\mathbb{A}-zI)x = KJ\varphi_- \\ \varphi_+ = \varphi_- - 2iK^*x \end{cases} \qquad (\mathrm{Im}\ \mathbb{A} = KJK^*),$$

or

$$\theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & E \end{pmatrix}.$$

In the system θ above $\mathbb{A}$ is a bounded linear operator, acting from $\mathfrak{H}_+$ into $\mathfrak{H}_-$, where $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$ is a rigged Hilbert space, $\mathbb{A} \supset T \supset A$, $\mathbb{A}^* \supset T^* \supset A$, A is a Hermitian operator in $\mathfrak{H}$, T is a non-Hermitian operator in $\mathfrak{H}$, K is a linear bounded operator from E into $\mathfrak{H}_-$, $J = J^* = J^{-1}$ is acting in E, $\varphi_\pm \in E$, φ_- is an input vector, φ_+ is an output vector, and $x \in \mathfrak{H}_+$ is a vector of the inner state of the system θ. The operator-valued function

$$W_\theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ \qquad (\varphi_+ = W_\theta(z)\varphi_-),$$

is the transfer operator-valued function of the system θ.

In [7] we established criteria for a given operator-valued R-function $V(z)$ to be realized in the form

$$V(z) = i[W_\theta(z) + I]^{-1}[W_\theta(z) - I]J.$$

It was shown that an operator-valued R-function

$$V(z) = Q + F \cdot z + \int_{-\infty}^{+\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) dG(t),$$

acting on a Hilbert space E ($\dim E < \infty$) with some invertibility condition can be realized if and only if

$$F = 0 \quad \text{and} \quad Qe = \int\limits_{-\infty}^{+\infty} \frac{t}{1+t^2}\, dG(t)e,$$

for all $e \in E$ such that

$$\int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty.$$

In terms of realizable operator-valued R-functions we specialize in subclasses of the following types:

(1) a subclass for which $\overline{\mathfrak{D}(A)} = \mathfrak{H}$, $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$
(2) a subclass for which $\overline{\mathfrak{D}(A)} \neq \mathfrak{H}$, $\mathfrak{D}(T) = \mathfrak{D}(T^*)$
(3) a subclass for which $\overline{\mathfrak{D}(A)} \neq \mathfrak{H}$, $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$

To prove the direct and inverse realization theorems for operator-valued R-functions in each subclass we build a functional model which generally speaking is an unbounded version of the Brodskiĭ-Livšic model with diagonal real part. This model for bounded linear operators was constructed in [11]. In the recent paper [4] the realization problems for contractive operator-valued functions are considered in terms of systems of the special kind. However, as it follows from [5], [7] not every contractive in the lower half-plane function can be realized by the Brodskiĭ-Livšic rigged operator colligation.

2. Preliminaries

Let $\mathfrak{H}$ denote a Hilbert space with inner product (x, y) and let A be a closed linear Hermitian operator, i.e. $(Ax, y) = (x, Ay)$ $(\forall x, y \in \mathfrak{D}(A))$, acting on a Hilbert space $\mathfrak{H}$ with generally speaking, non-dense domain $\mathfrak{D}(A)$. Let $\mathfrak{H}_0 = \overline{\mathfrak{D}(A)}$ and A^* be the adjoint to the operator A (we consider A as acting from $\mathfrak{H}_0$ into $\mathfrak{H}$).

We denote $\mathfrak{H}_+ = \mathfrak{D}(A^*)$ $((\overline{\mathfrak{D}(A^*)} = \mathfrak{H})$ with inner product

$$(f,g)_+ = (f,g) + (A^*f, A^*g) \qquad (f,g \in \mathfrak{H}_+) \tag{1}$$

and then construct the *rigged* Hilbert space $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$ [9], [7]. Here $\mathfrak{H}_-$ is the space of all linear functionals over $\mathfrak{H}_+$ that are continuous with respect to $\|\cdot\|_+$. The norms of these spaces are connected by the relations $\|x\| \leq \|x\|_+$ $(x \in \mathfrak{H}_+)$, $\|x\|_- \leq \|x\|$ $(x \in \mathfrak{H})$. The Riesz-Berezanskii operator (see [7]) $\mathcal{R}$ maps $\mathfrak{H}_-$ onto $\mathfrak{H}_+$ such that

$$\begin{aligned}(x,y)_- = (x,\mathcal{R}y) = (\mathcal{R}x,y) = (\mathcal{R}x,\mathcal{R}y)_+ \qquad (x,y\in\mathfrak{H}_-)\\ (u,v)_+ = (u,\mathcal{R}^{-1}v) = (\mathcal{R}^{-1}u,v) = (\mathcal{R}^{-1}x,\mathcal{R}^{-1}y)_- \qquad (u,v\in\mathfrak{H}_+)\end{aligned} \tag{2}$$

In what follows we use symbols $(+)$, $(\cdot)$, and $(-)$ to indicate the norms $\|\cdot\|_+$, $\|\cdot\|$, and $\|\cdot\|_-$ by which geometrical and topological concepts are defined in $\mathfrak{H}_+$, $\mathfrak{H}$, and $\mathfrak{H}_-$, respectively.

In the above settings $\mathfrak{D}(A) \subset \mathfrak{D}(A^*)(=\mathfrak{H}_+)$ and $A^*y = PAy$ $(\forall y \in \mathfrak{D}(A))$, where P is an orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_0$. Let

$$\mathfrak{L} := \mathfrak{H} \ominus \mathfrak{H}_0 \quad \mathfrak{M}_\lambda := (A-\lambda I)\mathfrak{D}(A) \quad \mathfrak{N}_\lambda := (\mathfrak{M}_{\bar\lambda})^\perp. \tag{3}$$

The subspace $\mathfrak{N}_\lambda$ is called a *defect subspace* of A for the point $\bar\lambda$. The cardinal number $\dim\mathfrak{N}_\lambda$ remains constant when λ is in the upper half-plane. Similarly, the number $\dim\mathfrak{N}_\lambda$ remains constant when λ is in the lower half-plane. The numbers $\dim\mathfrak{N}_\lambda$ and $\dim\mathfrak{N}_{\bar\lambda}$ $(\mathrm{Im}\lambda < 0)$ are called the *defect numbers* or *deficiency indices* of operator A [1]. The subspace $\mathfrak{N}_\lambda$ which lies in $\mathfrak{H}_+$ is the set of solutions of the equation $A^*g = \lambda Pg$.

Let now P_λ be the orthogonal projection onto $\mathfrak{N}_\lambda$, set

$$\mathfrak{B}_\lambda = P_\lambda\mathfrak{L}, \qquad \mathfrak{N}'_\lambda = \mathfrak{N}_\lambda \ominus \overline{\mathfrak{B}_\lambda} \tag{4}$$

It is easy to see that $\mathfrak{N}'_\lambda = \mathfrak{N}_\lambda \cap \mathfrak{H}_0$ and $\mathfrak{N}'_\lambda$ is the set of solutions of the equation $A^*g = \lambda g$ (see [27]), when $A^* : \mathfrak{H} \to \mathfrak{H}_0$ is the adjoint operator to A.

The subspace $\mathfrak{N}'_\lambda$ is the defect subspace of the densely defined Hermitian operator PA on $\mathfrak{H}_0$ (see [24]). The numbers $\dim\mathfrak{N}'_\lambda$ and $\dim\mathfrak{N}'_{\bar\lambda}$ $(\mathrm{Im}\lambda < 0)$ are called *semi-defect numbers* or the *semi-deficiency indices* of the operator A [19]. The von Neumann formula

$$\mathfrak{H}_+ = \mathfrak{D}(A^*) = \mathfrak{D}(A) + \mathfrak{N}_\lambda + \mathfrak{N}_{\bar\lambda}, \qquad (\mathrm{Im}\lambda \neq 0), \tag{5}$$

holds, but this decomposition is not direct for a non-densely defined operator A. There exists a generalization of von Neumann's formula [2], [26] to the case of a non-densely defined Hermitian operator (direct decomposition). We call an operator A *regular*, if PA is a closed operator in $\mathfrak{H}_0$. For a regular operator A we have

$$\mathfrak{H}_+ = \mathfrak{D}(A) + \mathfrak{N}'_\lambda + \mathfrak{N}'_{\bar\lambda} + \mathfrak{N}, \qquad (\mathrm{Im}\lambda \neq 0) \tag{6}$$

where $\mathfrak{N} := \mathcal{R}\mathfrak{L}$, $\mathcal{R}$ is the Riesz-Berezanskii operator. This is a generalization of von Neumann's formula. For $\lambda = \pm i$ we obtain the (+)-orthogonal decomposition

$$\mathfrak{H}_+ = \mathfrak{D}(A) \oplus \mathfrak{N}'_i \oplus \mathfrak{N}'_{-i} \oplus \mathfrak{N}. \tag{7}$$

Let $\tilde{A}$ be a closed Hermitian extension of the operator A. Then $\mathfrak{D}(\tilde{A}) \subset \mathfrak{H}_+$ and $P\tilde{A}x = A^*x$ $(\forall x \in \mathfrak{D}(\tilde{A}))$. According to [27] a closed Hermitian extension $\tilde{A}$ is said to be *regular* if $P\tilde{A}$ is closed. This implies that $\mathfrak{D}(\tilde{A})$ is (+)-closed. According to the theory of extensions of closed Hermitian operators A with non-dense domain [18], an operator U $(\mathfrak{D}(U) \subseteq \mathfrak{N}_i,\ \mathfrak{R}(U) \subseteq \mathfrak{N}_{-i})$ is called an *admissible operator* if $(U - I)f_i \in \mathfrak{D}(A)$ $(f_i \in \mathfrak{D}(U))$ only for $f_i = 0$. Then (see [3]) any symmetric extension $\tilde{A}$ of the non-densely defined closed Hermitian operator A, is defined by an isometric admissible operator U, $\mathfrak{D}(U) \subseteq \mathfrak{N}_i$, $\mathfrak{R}(U) \subseteq \mathfrak{N}_{-i}$ by the formula

$$\tilde{A}f_{\tilde{A}} = Af_A + (-if_i - iUf_i), \quad f_A \in \mathfrak{D}(A)$$

where $\mathfrak{D}(\tilde{A}) = \mathfrak{D}(A) \dot{+} (U - I)\mathfrak{D}(U)$. The operator $\tilde{A}$ is self-adjoint if and only if $\mathfrak{D}(U) = \mathfrak{N}_i$ and $\mathfrak{R}(U) = \mathfrak{N}_{-i}$.

Let us denote now by $P^+_{\mathfrak{N}}$ the orthogonal projection operator in $\mathfrak{H}_+$ onto $\mathfrak{N}$. We introduce a new inner product $(\cdot,\cdot)_1$ defined by

$$(f, g)_1 = (f, g)_+ + (P^+_{\mathfrak{N}} f, P^+_{\mathfrak{N}} g)_+ \tag{8}$$

for all $f, g \in \mathfrak{H}_+$. The obvious inequality

$$\|f\|^2_+ \leq \|f\|^2_1 \leq 2\|f\|^2_+$$

shows that the norms $\|\cdot\|_+$ and $\|\cdot\|_1$ are topologically equivalent. It is easy to see that the spaces $\mathfrak{D}(A)$, $\mathfrak{N}'_i$, $\mathfrak{N}'_{-i}$, $\mathfrak{N}$ are (1)-orthogonal. We write $\mathfrak{M}_1$ for the Hilbert space $\mathfrak{M} = \mathfrak{N}'_i \oplus \mathfrak{N}'_{-i} \oplus \mathfrak{N}$ with inner product $(f, g)_1$. We denote by $\mathfrak{H}_{+1}$ the space $\mathfrak{H}_+$ with norm $\|\cdot\|_1$, and by $\mathcal{R}_1$ the corresponding Riesz-Berezanskii operator related to the rigged Hilbert space $\mathfrak{H}_{+1} \subset \mathfrak{H} \subset \mathfrak{H}_{-1}$.

Denote by $[\mathfrak{H}_1, \mathfrak{H}_2]$ the set of all linear bounded operators acting from a Hilbert space $\mathfrak{H}_1$ into a Hilbert space $\mathfrak{H}_2$.

Definition. An operator $\mathbb{A} \in [\mathfrak{H}_+, \mathfrak{H}_-]$ is a *bi-extension* of A if both $\mathbb{A} \supset A$ and $\mathbb{A}^* \supset A$ hold.

If $\mathbb{A} = \mathbb{A}^*$, then $\mathbb{A}$ is called self-adjoint bi-extension of the operator A. It was mentioned in [7] that every self-adjoint bi-extension $\mathbb{A}$ of the regular Hermitian operator A is of the form:

$$\mathbb{A} = AP^+_{\mathfrak{D}(A)} + \left[A^* + \mathcal{R}_1^{-1}(S - \frac{i}{2}P^+_{\mathfrak{N}'_i} + \frac{i}{2}P^+_{\mathfrak{N}'_{-i}})\right] P^+_{\mathfrak{M}},$$

where S is an arbitrary (1)-self-adjoint operator in $[\mathfrak{M}_1, \mathfrak{M}_1]$. We write $\mathfrak{S}(A)$ for the class of bi-extensions of A. This class is closed in the weak topology and is invariant under taking adjoints (see [3], [27]).

Let $\mathbb{A}$ be a bi-extension of Hermitian operator A. The operator $\hat{A}f = \mathbb{A}f$, $\mathfrak{D}(\hat{A}) = \{f \in \mathfrak{H}, \mathbb{A}f \in \mathfrak{H}\}$ is called the *quasikernel* of $\mathbb{A}$. If $\mathbb{A} = \mathbb{A}^*$ and $\hat{A}$ is a quasi-kernel of $\mathbb{A}$ such that $A \neq \hat{A}$, $\hat{A}^* = \hat{A}$ then $\mathbb{A}$ is said to be a *strong* self-adjoint bi-extension of A.

Definition. We say that a closed densely defined linear operator T acting on a Hilbert space $\mathfrak{H}$ belongs to the class Ω_A if:

(1) $T \supset A$, $T^* \supset A$ where A is a closed Hermitian operator;
(2) $(-i)$ is a regular point of T.[1]

It was mentioned in [3] that lineals $\mathfrak{D}(T)$ and $\mathfrak{D}(T^*)$ are $(+)$-closed, the operators T and T^* are $(+, \cdot)$-bounded. The following theorem [27] is an analogue to von Neumann's formulae for the class Ω_A.

Theorem 1. *If an operator T belongs to the class Ω_A, then*

$$\begin{cases} \mathfrak{D}(T) = \mathfrak{D}(A) \dotplus (I - \Phi)\mathfrak{N}_i \\ \mathfrak{D}(T^*) = \mathfrak{D}(A) \dotplus (\Phi^* - I)\mathfrak{N}_{-i} \end{cases}$$

where Φ and Φ^* are admissible operators in $[\mathfrak{N}_i, \mathfrak{N}_{-i}]$ and $[\mathfrak{N}_{-i}, \mathfrak{N}_i]$ respectively.

There is a modification of the last theorem [27], [28].

Theorem 2. *I. For each operator of the class Ω_A there exists an operator M on the space $\mathfrak{M}_1$ with the following properties:*

(1) $\mathfrak{D}(M) = \mathfrak{N}'_i \oplus \mathfrak{N}$ and $\mathfrak{R}(M) = \mathfrak{N}'_{-i} \oplus \mathfrak{N}$;
(2) $Mx + x = 0$ only for $x = 0$, and $M^*x + x = 0$ only for $x = 0$. Moreover, the following hold:

$$\begin{cases} \mathfrak{D}(T) = \mathfrak{D}(A) \oplus (M + I)(\mathfrak{N}'_i \oplus \mathfrak{N}) \\ \mathfrak{D}(T^*) = \mathfrak{D}(A) \oplus (M^* + I)(\mathfrak{N}'_{-i} \oplus \mathfrak{N}) \end{cases} \tag{9}$$

II. Conversely, for each pair of (1)-adjoint operators M and M^* in $[\mathfrak{M}_1, \mathfrak{M}_1]$ with the properties (1) and (2) formulas (9) give a corresponding operator T in class Ω_A. Moreover, if $f = g + (M + I)\varphi$, $g \in \mathfrak{D}(A)$, and $\varphi \in \mathfrak{N}'_i \oplus \mathfrak{N}$, then

$$Tf = Ag + A^*(I + M)\varphi + i\mathcal{R}_1^{-1}P^+_{\mathfrak{N}}(I - M)\varphi \quad (f \in \mathfrak{D}(T)), \tag{10}$$

Similarly, if $f = g + (M^* + I)\psi$, $g \in \mathfrak{D}(A)$, and $\psi \in \mathfrak{N}'_{-i} \oplus \mathfrak{N}$, then

$$T^*f = Ag + A^*(I + M^*)\psi + i\mathcal{R}_1^{-1}P^+_{\mathfrak{N}}(M^* - I)\psi \quad (f \in \mathfrak{D}(T)), \tag{11}$$

The following theorems can be found in [27],[28].

[1] The condition, that $(-i)$ is a regular point in the definition of the class Ω_A is not essential. It is sufficient to require the existence of some regular point for T.

Theorem 3. *Let T be an operator of Ω_A class such that A is the maximal Hermitian part of T and T^*. Let M be the corresponding operator from the Theorem 2 with the properties* (1) *and* (2)*. Then the operators $MM^* - I$ and $M^*M - I$ are invertible in $\mathfrak{M}$.*

Definition. A regular operator A is called *O-operator* if its semidefect numbers (defect numbers of an operator PA) are equal to zero.

Theorem 4. *Let T be an operator of the class Ω_A where A is a regular Hermitian operator. Then the following statements are valid:*

(1) *If A is an O-operator then*

$$\mathfrak{D}(T) = \mathfrak{D}(T^*) = \mathfrak{H}_+$$

and the operator $T - T^*$ is $(\cdot,\cdot)$-continuous.

(2) *If A is not an O-operator then either $\mathfrak{D}(T)$ or $\mathfrak{D}(T^*)$ does not coincide with $\mathfrak{H}_+$.*

Proof. Since T is an operator of the class Ω_A then $\mathfrak{D}(T)$ and $\mathfrak{D}(T^*)$ are subspaces of $\mathfrak{H}_+$. Let M and M^* be the operators defined in the Theorem 2. In this case $\mathfrak{D}(M) = \mathfrak{N}'_i \oplus \mathfrak{N}$, $\mathfrak{R}(M) \subseteq \mathfrak{N}'_{-i} \oplus \mathfrak{N}$, $\mathfrak{D}(M^*) = \mathfrak{N}'_{-i} \oplus \mathfrak{N}$, and $\mathfrak{R}(M^*) \subseteq \mathfrak{N}'_i \oplus \mathfrak{N}$. Formulas (9) imply that $\mathfrak{R}(M+I)$ and $\mathfrak{R}(M^*+I)$ are $(+)$- and (1)-subspaces as well. Consider the (1)-orthogonal complements

$$[\mathfrak{R}(M+I)]^{\perp} \qquad \text{and} \qquad [\mathfrak{R}(M^*+I)]^{\perp}.$$

Let us assume that A is not an O-operator. Then the semidefect numbers of A are not both zero. For any $y \in [\mathfrak{R}(M^*+I)]^{\perp}$ and for any $x \in \mathfrak{N}'_{-i} \oplus \mathfrak{N}$ we have

$$((M^*+I)x, y)_1 = 0.$$

Furthermore, using (1)-orthogonality relation one can show that

$$\begin{aligned}
((M^*+I)x, y)_1 &= \left((M^*+I)x, P^+_{\mathfrak{N}'_i} y + P^+_{\mathfrak{N}'_{-i}} y + P^+_{\mathfrak{N}} y\right)_1 \\
&= \left(M^*x, P^+_{\mathfrak{N}'_i} y + P^+_{\mathfrak{N}'_{-i}} y + P^+_{\mathfrak{N}} y\right)_1 + \left(x, P^+_{\mathfrak{N}'_i} y + P^+_{\mathfrak{N}'_{-i}} y + P^+_{\mathfrak{N}} y\right)_1 \\
&= \left(M^*x, P^+_{\mathfrak{N}'_i} y + P^+_{\mathfrak{N}} y\right)_1 + \left(M^*x, P^+_{\mathfrak{N}'_{-i}} y\right)_1 + \left(x, P^+_{\mathfrak{N}'_{-i}} y + P^+_{\mathfrak{N}} y\right)_1 \\
&\quad + \left(x, P^+_{\mathfrak{N}'_i} y\right)_1 \\
&= \left(x, M(P^+_{\mathfrak{N}'_i} + P^+_{\mathfrak{N}})y\right)_1 + \left(x, (P^+_{\mathfrak{N}'_{-i}} + P^+_{\mathfrak{N}})y\right)_1 \\
&= 0.
\end{aligned}$$

Therefore, since M maps $\mathfrak{N}'_i \oplus \mathfrak{N}$ into $\mathfrak{N}'_{-i} \oplus \mathfrak{N}$ we have that

$$M(P^+_{\mathfrak{N}'_i} + P^+_{\mathfrak{N}})y = -(P^+_{\mathfrak{N}'_{-i}} + P^+_{\mathfrak{N}})y, \tag{12}$$

Let us denote $z=(P^+_{\mathfrak{N}'_i}+P^+_{\mathfrak{N}})y$. Then, obviously,

$$P^+_{\mathfrak{N}}(M+I)z=0, \qquad (z\in\mathfrak{N}'_i\oplus\mathfrak{N}), \tag{13}$$

Hence, if $y\in[\mathfrak{R}(M^*+I)]^\perp$ then

$$z=(P^+_{\mathfrak{N}'_i}+P^+_{\mathfrak{N}})y\in\operatorname{Ker}\left[P^+_{\mathfrak{N}}(M+I)z\right] \quad \text{and} \quad y=z-P^+_{\mathfrak{N}'_{-i}}Mz.$$

Let now $z\in\operatorname{Ker}\left[P^+_{\mathfrak{N}}(M+I)\right]$. We show that the vector $y=z-P^+_{\mathfrak{N}'_{-i}}Mz$ belongs to $[\mathfrak{R}(M^*+I)]^\perp$. To do that it is sufficient to show that for indicated vector y the relation (12) holds. Indeed,

$$\begin{aligned}-(P^+_{\mathfrak{N}'_{-i}}+P^+_{\mathfrak{N}})y&=-P^+_{\mathfrak{N}}z+P^+_{\mathfrak{N}'_{-i}}Mz=P^+_{\mathfrak{N}}Mz+P^+_{\mathfrak{N}'_{-i}}Mz\\&=Mz=M(P^+_{\mathfrak{N}'_i}+P^+_{\mathfrak{N}})y.\end{aligned}$$

Hence,

$$[\mathfrak{R}(M^*+I)]^\perp=(I-P^+_{\mathfrak{N}'_i}M)\{\operatorname{Ker}[P^+_{\mathfrak{N}}(M+I)]\}. \tag{14}$$

It can be shown similarly, that

$$[\mathfrak{R}(M+I)]^\perp=(I-P^+_{\mathfrak{N}'_i}M)\{\operatorname{Ker}[P^+_{\mathfrak{N}}(M^*+I)]\}. \tag{15}$$

Let us assume that $[\mathfrak{R}(M^*+I)]^\perp=0$. It is easy to see that equality $(I-P^+_{\mathfrak{N}'_i}M)z=0$ implies that if $z=0$ then $\operatorname{Ker}[P^+_{\mathfrak{N}}(M^*+I)]=0$. Then operator (M^*+I) maps $\mathfrak{N}'_{-i}\oplus\mathfrak{N}$ onto $\mathfrak{M}$. Therefore, there exists vector $x\neq 0$, $x\in\mathfrak{N}'_{-i}\oplus\mathfrak{N}$ such that $P^+_{\mathfrak{N}}(M^*+I)x=0$ and so $\operatorname{Ker}[P^+_{\mathfrak{N}}(M^*+I)]\neq 0$. Thus, $[\mathfrak{R}(M+I)]^\perp\neq 0$. Together with formulas (9) that proves the first part of the theorem.

Let now A be a regular O-operator, i.e. $\mathfrak{N}'_i=\mathfrak{N}'_{-i}=\{0\}$ and consequently $\mathfrak{M}=\mathfrak{N}$. Let us assume that x is (+)-orthogonal to $\mathfrak{D}(T)$. According to the formulas (9) x is (+)-orthogonal to $\mathfrak{D}(A)$ and therefore belongs to $\mathfrak{N}$. On the other hand (9) imply that x is (+)-orthogonal to $(M+I)\mathfrak{N}$. Hence, $(M^*+I)x=0$. Using Theorem 1 we conclude that $x=0$. Therefore, $\mathfrak{D}(T)$ is (+)-dense in $\mathfrak{H}_+$. In the same way one can prove that $\mathfrak{D}(T^*)$ is (+)-dense in $\mathfrak{H}_+$.

Definition. An operator $\mathbb{A}$ in $[\mathfrak{H}_+,\mathfrak{H}_-]$ is called a $(*)$-extension of an operator T of the class Ω_A if both $\mathbb{A}\supset T$ and $\mathbb{A}^*\supset T^*$.

This $(*)$-extension is called *correct*, if an operator $\mathbb{A}_R=\frac{1}{2}(\mathbb{A}+\mathbb{A}^*)$ is a strong self-adjoint bi-extension of an operator A. It is easy to show that if $\mathbb{A}$ is a $(*)$-extension of T, the T and T^* are quasi-kernels of $\mathbb{A}$ and $\mathbb{A}^*$, respectively.

Definition. We say the operator T of the class Ω_A belongs to the class Λ_A if

(1) T admits a correct $(*)$-extension;
(2) A is a maximal common Hermitian part of T and T^*.

The following theorem can be found in [28].

Theorem 5. *Let an operator T belong to Ω_A and M be an operator in $[\mathfrak{M}, \mathfrak{M}]$ that is related to T by Theorem 2. Then T belongs to Λ_A if and only if there exists either (1)-isometric operator or $(\cdot)$-isometric operator U in $[\mathfrak{N}'_i, \mathfrak{N}'_{-i}]$ such that*

$$\left\{ \begin{array}{l} (U+I)\mathfrak{N}'_i + (M+I)(\mathfrak{N}'_i \oplus \mathfrak{N}) = \mathfrak{M}, \\ (U+I)\mathfrak{N}'_i + (M+I)(\mathfrak{N}'_i \oplus \mathfrak{N}) = \mathfrak{M}. \end{array} \right. \tag{16}$$

Corollary 1. *If a closed Hermitian operator A has finite and equal defect indices then the class Ω_A coincides with the class Λ_A.*

Let A be a closed Hermitian operator on $\mathfrak{H}$ and $\mathfrak{h}$ be a Hilbert space such that $\mathfrak{H}$ is a subspace of $\mathfrak{h}$. Let $\tilde{A}$ be a self-adjoint extension of A on $\mathfrak{h}$, and $\tilde{E}(t)$ be the spectral function of $\tilde{A}$. An operator function $R_\lambda = P_{\mathfrak{H}}(\tilde{A} - \lambda I)^{-1}|_{\mathfrak{H}}$ is called a *generalized resolvent* of A, and $E(t) = P_{\mathfrak{H}}\tilde{E}(t)|_{\mathfrak{H}}$ is the corresponding *generalized spectral function.* Here

$$R_\lambda = \int\limits_{-\infty}^{\infty} \frac{dE(t)}{t-\lambda} \quad (\mathrm{Im}\lambda \neq 0). \tag{17}$$

If $\mathfrak{h} = \mathfrak{H}$ then R_λ and $E(t)$ are called *canonical resolvent* and *canonical spectral function*, respectively. According to [21] we denote by $\hat{R}_\lambda$ the $(-, \cdot)$-continuous operator from $\mathfrak{H}_-$ into $\mathfrak{H}$ which is adjoint to $R_{\bar{\lambda}}$:

$$(\hat{R}_\lambda f, g) = (f, R_{\bar{\lambda}} g) \quad (f \in \mathfrak{H}_-,\, g \in \mathfrak{H}). \tag{18}$$

It follows that $\hat{R}_\lambda f = R_\lambda f$ for $f \in \mathfrak{H}$, so that $\hat{R}_\lambda$ is an extension of R_λ from $\mathfrak{H}$ to $\mathfrak{H}_-$ with respect to $(-, \cdot)$-continuity. The function $\hat{R}_\lambda$ of the parameter λ, $(\mathrm{Im}\lambda \neq 0)$ is called the *extended generalized (canonical) resolvent* of the operator A. We write $\aleph$ to denote the family of all finite intervals on the real axis. It is known [21] that if $\mathit{\Delta} \in \aleph$ then $E(\mathit{\Delta})\mathfrak{H} \subset \mathfrak{H}_+$ and the operator $E(\mathit{\Delta})$ is $(\cdot, +)$-continuous. We denote by $\hat{E}(\mathit{\Delta})$ the $(-, \cdot)$-continuous operator from $\mathfrak{H}_-$ to $\mathfrak{H}$ that is adjoint to $E(\mathit{\Delta}) \in [\mathfrak{H}, \mathfrak{H}_+]$. Similarly,

$$(\hat{E}(\mathit{\Delta}) f, g) = (f, E(\mathit{\Delta}) g) \quad (f \in \mathfrak{H}_-,\, g \in \mathfrak{H}) \tag{19}$$

One can easily see that $\hat{E}(\mathit{\Delta})f = E(\mathit{\Delta})f$, $\forall f \in \mathfrak{H}$, so that $\hat{E}(\mathit{\Delta})$ is the extension of $E(\mathit{\Delta})$ by continuity. We say that $\hat{E}(\mathit{\Delta})$, as a function of $\mathit{\Delta} \in \aleph$, is the *extended generalized (canonical) spectral function* of A corresponding to the self-adjoint extension $\tilde{A}$ (or to the original spectral function $E(\varDelta)$). It is known [21] that $\hat{E}(\varDelta) \in [\mathfrak{H}_-, \mathfrak{H}_+]$, $\forall \varDelta \in \aleph$, and $(\hat{E}(\varDelta)f, f) \geq 0$ for all $f \in \mathfrak{H}_-$. It is also known [21] that the complex scalar measure $(E(\varDelta)f, g)$ is a complex function of bounded variation on the real axis. However, $(\hat{E}(\varDelta)f, g)$ may be unbounded for $f, g \in \mathfrak{H}_-$.

Now let $\hat{R}_\lambda$ be an extended generalized (canonical) resolvent of a closed Hermitian operator A and let $\hat{E}(\varDelta)$ be the corresponding extended generalized (canonical) spectral function. It was shown in [21] that for any $f, g \in \mathfrak{H}_-$,

$$\int_{-\infty}^{+\infty} \frac{|d(\hat{E}(\varDelta)f, g)|}{1+t^2} < \infty \tag{20}$$

and the following integral representation holds

$$\hat{R}_\lambda - \frac{\hat{R}_i + \hat{R}_{-i}}{2} = \int_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right) d\hat{E}(t) \tag{21}$$

Lemma 6. ([1],[7]) *Let $\mathbb{A} = AP^+_{\mathfrak{D}(A)} + [A^* + \mathcal{R}_1^{-1}(S - \frac{i}{2}P^+_{\mathfrak{N}'_i} + \frac{i}{2}P^+_{\mathfrak{N}'_{-i}})]P^+_{\mathfrak{M}}$ be a strong self-adjoint bi-extension of a regular Hermitian operator A with the quasi-kernel $\hat{A}$ and let $\hat{E}(\varDelta)$ be the extended generalized (canonical) spectral function of $\hat{A}$. Then for every $f \in \mathfrak{H} \oplus L$, $f \neq 0$, and for every $g \in \mathfrak{H}_-$ there is an integral representation*

$$(\bar{R}_\lambda f, g) = \int_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right) d(\hat{E}(t)f, g) + \frac{1}{2}((\hat{R}_i + \hat{R}_{-i})f, g). \tag{22}$$

Here $L = \mathfrak{R}\Big[\mathcal{R}_1^{-1}(P^+_{\mathfrak{M}}S - \frac{i}{2}P^+_{\mathfrak{N}'_i} + \frac{i}{2}P^+_{\mathfrak{N}'_{-i}})\Big]$, $\bar{R}_\lambda = \overline{(\mathbb{A} - \lambda I)^{-1}}$.

Theorem 7. ([7]) *Let $\mathbb{A} = AP^+_{\mathfrak{D}(A)} + [A^* + \mathcal{R}_1^{-1}(S - \frac{i}{2}P^+_{\mathfrak{N}'_i} + \frac{i}{2}P^+_{\mathfrak{N}'_{-i}})]P^+_{\mathfrak{M}}$ be a strong self-adjoint bi-extension of a regular Hermitian operator A with the quasi-kernel $\hat{A}$ and let $\hat{E}(\varDelta)$ be the generalized (canonical) spectral function of $\hat{A}$, $F = \mathfrak{H}_+ \ominus \mathfrak{D}(\hat{A})$, $L = \mathcal{R}_1^{-1}(P^+_{\mathfrak{M}}S - \frac{i}{2}P^+_{\mathfrak{N}'_i} + \frac{i}{2}P^+_{\mathfrak{N}'_{-i}})F$. Then for any $f \in L \dotplus \mathfrak{L}$, $f \neq 0$,*

$$\int_{-\infty}^{+\infty} d(\hat{E}(t)f, f) = \infty, \qquad \textit{if} \quad f \notin \mathfrak{L}, \tag{23}$$

and

$$\int_{-\infty}^{+\infty} d(\hat{E}(t)f, f) < \infty, \qquad \textit{if} \quad f \in \mathfrak{L}. \tag{24}$$

Moreover, there exist real constants b and c such that

$$c\|f\|_-^2 \leq \int\limits_{-\infty}^{+\infty} \frac{d(\hat{E}(t)f,f)}{1+t^2} \leq b\|f\|_-^2 \tag{25}$$

for all $f \in L \dotplus \mathfrak{L}$.

In a weaker form Theorem 7 also appears at [1]. We briefly sketch the proof of this theorem.

Proof. Let us choose a point z with $\mathrm{Im} z \neq 0$ to be a regular point of the operator $\mathbb{A}$ and consider function $f(z)$ defined for all $f \in L \dotplus \mathfrak{L}$ by the formula:

$$f(z) = \big((\mathbb{A} - zI)^{-1}f, f\big).$$

It can be seen that $f(z) = \overline{f(\bar{z})}$ and

$$\mathrm{Im}\, f(z) = \mathrm{Im}\, z \left\|(\mathbb{A} - \bar{z}I)^{-1}f\right\|^2,$$

which means that $f(z)$ is an analytic R-function (see [17]) and according to the Lemma 6 has the integral representation

$$f(z) = \int\limits_{-\infty}^{+\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\big(\hat{E}(t)f,f\big) + \frac{1}{2}\big((\hat{R}_i + \hat{R}_{-i})f,f)\big).$$

This representation implies that

$$\lim_{\eta\to\infty} \frac{\mathrm{Im}\, f(i\eta)}{\eta} = 0,$$

and therefore (see [13], [17])

$$\sup_{\eta>0} \eta \mathrm{Im}\, f(i\eta) = \int\limits_{-\infty}^{+\infty} d(\hat{E}(t)f,f).$$

Now, let us pick $f \in L \dotplus \mathfrak{L}$ such that $f \notin \mathfrak{L}$ and show that in this case

$$\sup_{\eta>0} \eta \mathrm{Im}\, f(i\eta) = \infty.$$

It can be shown that for any $f \in L \dotplus \mathfrak{L}$ there are $x_0(i\eta) \in \mathfrak{D}(\hat{A})$, $f_1 \in F$ and $f_2 \in \mathfrak{L}$ such that

$$(\mathbb{A} + i\eta I)^{-1}f = x_0(i\eta) + f_1,$$

$$x_0(i\eta) = (\hat{A} + i\eta I)^{-1}\big[f_2 - \mathcal{R}_1 P_{\mathfrak{N}} S f_1 - (\mathbb{A}^* + i\eta I) f_1\big],$$

or

$$i\eta(x_0(i\eta) + f_1) = -\hat{A} x_0(i\eta) - A^* f_1 + f_2 - \mathcal{R}_1 P_{\mathfrak{N}} S f_1.$$

The vectors $x_0(i\eta)$ and f_1 are $(+1)$-orthogonal and hence

$$\|(\mathbb{A} + i\eta I)^{-1} f\|_+^2 = \|x_0(i\eta)\|_+^2 + \|f_1\|_+^2.$$

If we assume that the number set $\{\|x_0(i\eta)\|_+^2\}$ is bounded, i.e.

$$\sup\{\|x_0(i\eta)\|_+\} = c < \infty,$$

then, due to the $(+,\cdot)$-continuity of the operator $\hat{A}$ (see [28]), there exists a constant $d > 0$, such that for all $x_0(i\eta) \in \mathfrak{D}(\hat{A})$

$$\|\hat{A} x_0(i\eta)\| \leq d\|x_0(i\eta)\|_+ \leq d\sqrt{c},$$

and

$$\begin{aligned} \|x_0(i\eta) + f_1\| &\leq \frac{1}{\eta}\|f_2 - \hat{A} x_0(i\eta) - A^* f_1\| \\ &\leq \frac{1}{\eta}\left(d\sqrt{c} + \|f_2\| + \|A^* f_1\|\right). \end{aligned}$$

This implies $\lim\limits_{\eta\to\infty} x_0(i\eta) = -f_1$. The set $\{x_0(i\eta)\}$ is bounded in $\mathfrak{H}_+$ and therefore weakly compact. This means there exists such an element $x_0 \in \mathfrak{H}_+$ that

$$\lim_{\eta_n\to\infty} (x_0(i\eta_n), \varphi) = (x_0, \varphi), \quad \forall \varphi \in \mathfrak{H}_-,$$

where $\{x_0(i\eta_n)\}$ is a sequence of the elements of the set $\{x_0(i\eta)\}$ and $x_0 \in \mathfrak{H}_+$. Thus $x_0 = -f_1$. On the other hand

$$\mathfrak{D}(\hat{A}) = \mathfrak{D}(A) \oplus \operatorname{Ker}\left[P^+_{\mathfrak{M}} S - \frac{i}{2} P_{\mathfrak{N}'_i} + \frac{i}{2} P_{\mathfrak{N}'_{-i}}\right],$$

is a subspace in $\mathfrak{H}_+$ and must be weakly closed providing $x_0 \in \mathfrak{D}(\hat{A})$. Considering the fact that $f_1 \in F$, $F = \mathfrak{H}_+ \ominus \mathfrak{D}(\hat{A})$, and $x_0 = -f_1$ we obtain a contradiction. Hence for all $f \in L \dotplus \mathfrak{L}$, $f \notin \mathfrak{L}$

$$\int_{-\infty}^{+\infty} d(\hat{E}(t) f, f) = \sup_{\eta>0} \eta \operatorname{Im} f(i\eta) = \infty.$$

To prove relation (23) we assume that $f \in \mathfrak{L}$. In this case $f_1 = 0$ and $(\mathbb{A} + i\eta I)^{-1} f = x_0(i\eta)$. The latter yields

$$\|(\hat{A} + i\eta I) x_0(i\eta)\|^2 = \|f\|^2.$$

Further it is not hard to get the inequality

$$\eta^2\|x_0)i\eta)\|^2 \le \|(\hat{A}+i\eta I)x_0(i\eta)\|^2 = \|f\|^2,$$

that implies

$$\eta \mathrm{Im}\, f(i\eta) = \eta^2\|(\mathbb{A}+i\eta I)^{-1}f\| \le \|f\|^2 < \infty.$$

The last inequality proves (24).

It can be shown that $(\mathbb{A}+iI)^{-1} \in \mathfrak{N}_{-i}$ for all $f \in L \dot{+} \mathfrak{L}$. The norms $\|\cdot\|$ and $\|\cdot\|_+$ are equivalent on $\mathfrak{N}_{\pm i}$ and so are the norms $\|\cdot\|$ and $\|\cdot\|_-$ (see [28]). Therefore

$$c\|f\|_-^2 \le \mathrm{Im}\, f(i) \le b\|f\|_-^2, \quad b>0,\, c>0 - \text{const}.$$

Combining this with

$$\mathrm{Im}\, f(i) = \frac{1}{2i}\big((\bar{R}_i - \bar{R}_{-i})f,f\big) = \int\limits_{-\infty}^{+\infty} \frac{d(\hat{E}(t)f,f)}{1+t^2},$$

we obtain the relation (25).

Corollary 2. *In the settings of Theorem 7 for all $f,g \in L \dot{+} \mathfrak{L}$*

$$\left|\left(\frac{\hat{R}_i+\hat{R}_{-i}}{2}f,g\right)\right| \le a\sqrt{\int_{-\infty}^{+\infty}\frac{d(\hat{E}(t)f,f)}{1+t^2}}\cdot\sqrt{\int_{-\infty}^{+\infty}\frac{d(\hat{E}(t)g,g)}{1+t^2}}, \tag{26}$$

where $a>0$ is a constant (see [1]).

3. Linear Stationary Conservative Dynamic Systems

In this section we consider linear stationary conservative dynamic systems (l. s. c. d. s.) θ of the form

$$\begin{cases} (\mathbb{A}-zI) = KJ\varphi_- \\ \varphi_+ = \varphi_- - 2iK^*x \end{cases} \qquad (\mathrm{Im}\ \mathbb{A} = KJK^*)\,. \tag{27}$$

In a system θ of the form (27) $\mathbb{A}$, K and J are bounded linear operators in Hilbert spaces, φ_- is an input vector, φ_+ is an output vector, x is an inner state vector of the system θ. For our purposes we need the following more precise definition:

Definition. *The array*

$$\theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & E \end{pmatrix} \tag{28}$$

is called a linear stationary conservative dynamic system (l.s.c.d.s.) or Brodskiĭ-Livšic rigged operator colligation if

1. $\mathbb{A}$ *is a correct $(*)$-extension of an operator T of the class Λ_A.*
2. $J = J^* = J^{-1} \in [E,E], \quad \dim E < \infty$
3. $\mathbb{A} - \mathbb{A}^* = 2iKJK^*$, *where $K \in [E,\mathfrak{H}_-] \qquad (K^* \in [\mathfrak{H}_+,E])$*

In this case, the operator K is called a *channel operator* and J is called a *direction operator* [10], [20]. A system θ of the form (30) will be called a *scattering* system (*dissipative* operator colligation) if $J = I$. We will associate with the system θ an operator-valued function

$$W_\theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ \tag{29}$$

which is called *a transfer operator-valued function* of the system θ or a characteristic operator-valued function of Brodskiĭ-Livšic rigged operator colligations. It may be shown [10], that the transfer operator-function of the system θ of the form (28) has the following properties:

$$\begin{aligned} &W_\theta^*(z)JW_\theta(z) - J \geq 0 \quad (\mathrm{Im}\ z > 0, z \in \rho(T)) \\ &W_\theta^*(z)JW_\theta(z) - J = 0 \quad (\mathrm{Im}\ z = 0, z \in \rho(T)) \\ &W_\theta^*(z)JW_\theta(z) - J \leq 0 \quad (\mathrm{Im}\ z < 0, z \in \rho(T)) \end{aligned} \tag{30}$$

where $\rho(T)$ is the set of regular points of an operator T. Similar relations take place if we change $W_\theta(z)$ to $W_\theta^*(z)$ in (30). Thus, a transfer operator-valued function of the system θ of the form (28) is J-contractive in the lower half-plane on the set of regular points of an operator T and J-unitary on real regular points of an operator T.

Let θ be a l. s. c. d. s. of the form (28). We consider an operator-valued function

$$V_\theta(z) = K^*(\mathbb{A}_R - zI)^{-1}K. \tag{31}$$

The transfer operator-function $W_\theta(z)$ of the system θ and an operator-function $V_\theta(z)$ of the form (31) are connected by the relation

$$V_\theta(z) = i[W_\theta(z) + I]^{-1}[W_\theta(z) - I]J \tag{32}$$

As it is known [1] an operator-function $V(z) \in [E, E]$ is called an *operator-valued R-function* if it is holomorphic in the upper half-plane and $\mathrm{Im}\ V(z) \geq 0$ when $\mathrm{Im}\ z > 0$.

It is known [17], [22], [27] that an operator-valued R-function acting on a Hilbert space E ($\dim E < \infty$) has an integral representation

$$V(z) = Q + F \cdot z + \int_{-\infty}^{+\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) dG(t), \tag{33}$$

where $Q = Q^*$, $F \geq 0$ in the Hilbert space E, $G(t)$ is non-decreasing operator-function on $(-\infty, +\infty)$ for which

$$\int_{-\infty}^{+\infty} \frac{dG(t)}{1+t^2} \in [E, E].$$

Definition. *We call an operator-valued R-function acting on a Hilbert space E $(\dim E < \infty)$ realizable if in some neighborhood of the point $(-i), the function V(z)$ can be represented in the form*

$$V(z) = i[W_\theta(z) + I]^{-1}[W_\theta(z) - I]J \tag{34}$$

where $W_\theta(z)$ is a transfer operator-function of some l.s.c.d.s. θ with the direction operator J $(J = J^* = J^{-1} \in [E, E])$.

Definition. *An operator-valued R-function $V(z) \in [E, E]$ $\quad(\dim E < \infty)$ will be said to be a member of the class $N(R)$ if in the representation* (33) *we have*

$$\begin{aligned} &i) \qquad F = 0, \\ &ii) \qquad Qe = \int_{-\infty}^{+\infty} \frac{t}{1+t^2} dG(t)e \end{aligned}$$

for all $e \in E$ such that

$$\int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty.$$

The next result is proved in [7].

Theorem 8. *Let θ be a l.s.c.d.s. of the form* (28), $\dim E < \infty$. *Then the operator-function $V_\theta(z)$ of the form* (31), (32) *belongs to the class $N(R)$.*

The following converse result was also established in [7].[2]

Theorem 9. *Suppose that the operator-valued function $V(z)$ is acting on a finite-dimensional Hilbert space E and belong to the class $N(R)$. Then $V(z)$ admits a realization by the system θ of the form* (28) *with a preassigned direction operator J for which $I+iV(-i)J$ is invertible.*

Remark. It was mentioned in [7] that when $J = I$ the invertibility condition for $I+iV(\lambda)J$ is satisfied automatically.

Now we are going to introduce three distinct subclasses of the class of realizable operator-valued functions $N(R)$.

Definition. *An operator-valued R-function $V(z) \in [E, E]$ $\quad(\dim E < \infty)$ of the class $N(R)$ is said to be a member of the subclass $N_0(R)$ if in the representation (33)*

$$\int_{-\infty}^{+\infty} (dG(t)e, e)_E = \infty, \qquad (e \in E, e \neq 0).$$

[2] The method of rigged Hilbert spaces for the solving of inverse problems of the theory of characteristic operator-valued functions was introduced in [25] and developed further in [1].

Consequently, the operator-function $V(z)$ of the class $N_0(R)$ has the representation

$$V(z) = Q + \int_{-\infty}^{+\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) dG(t), \qquad (Q = Q^*). \tag{35}$$

Note, that the operator Q can be an arbitrary self-adjoint operator on the Hilbert space E.

Definition. *An operator-valued R-function $V(z) \in [E, E]$ $(\dim E < \infty)$ of the class $N(R)$ is said to be a member of the subclass $N_1(R)$ if in the representation* (33)

$$\int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty, \qquad (e \in E). \tag{36}$$

It is easy to see that the operator-valued function $V(z)$ of the class $N_1(R)$ has a representation

$$V(z) = \int_{-\infty}^{+\infty} \frac{1}{t-z}\, dG(t) \tag{37}$$

Definition. *An operator-valued R-function $V(z) \in [E, E]$, $\quad (\dim E < \infty)$ of the class $N(R)$ is said to be a member of the subclass $N_{01}(R)$ if the subspace*

$$E_\infty = \left\{ e \in E : \int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty \right\}$$

possesses a property: $E_\infty \neq \emptyset$, $\quad E_\infty \neq E$.

One may notice that $N(R)$ is a union of three distinct subclasses $N_0(R)$, $N_1(R)$ and $N_{01}(R)$. The following theorem is an analogue of the Theorem 8 for the class $N_0(R)$.

Theorem 10. *Let θ be a l. s. c. d. s. of the form (28), $\dim E < \infty$ where A is a linear closed Hermitian operator with dense domain and $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$. Then the operator-valued function $V_\theta(z)$ of the form* (31), (32) *belongs to the class $N_0(R)$.*

Proof. Relying on Theorem 8 an operator-valued function $V_\theta(z)$ of the system θ mentioned in the statement belongs to the class $N(R)$. Since $N_0(R)$ is a subclass of $N(R)$, it is sufficient to show that

$$\int_{-\infty}^{+\infty} (dG(t)e, e)_E = \infty, \qquad (e \in E, e \neq 0).$$

According to Theorem 7, if for some vector $f \in E$ we have that $Kf \notin \mathfrak{L}$ where $\mathfrak{L} = \mathfrak{H} \ominus \overline{\mathfrak{D}(A)}$, then

$$\int_{-\infty}^{+\infty} (dG(t)f, f)_E = \infty, \quad \text{where } G(t) = K^* E(t) K, \tag{38}$$

$E(t)$ is an extended generalized spectral function of the operator $\hat{A}$. Here $\hat{A}$ is the quasi-kernel of an operator

$$\mathbb{A}_R = \frac{\mathbb{A} + \mathbb{A}^*}{2}.$$

It is given that A is a closed Hermitian operator with dense domain ($\overline{\mathfrak{D}(A)} = \mathfrak{H}$), which implies that $\mathfrak{L} = \emptyset$. Thus, for any $f \in E$ such that $f \neq 0$ we have

$$Kf \notin \mathfrak{L},$$

and (38) holds. Therefore, $V_\theta(z)$ belongs to the class $N_0(R)$.

Note that the condition (38) has also appeared in [14], [15]. Theorem 11 below is a version of the Theorem 9 for the class $N_0(R)$.

Theorem 11. *Let an operator-valued function $V(z)$ acting on a finite-dimensional Hilbert space E belong to the class $N_0(R)$. Then it admits a realization by the system θ of the form* (28) *with a preassigned directional operator J for which $I + iV(-i)J$ is invertible, densely defined closed Hermitian operator A, and $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$.*

Proof. Since $N_0(R)$ is a subclass of $N(R)$ then all conditions of Theorem 9 are satisfied and operator-valued function $V(z) \in N_0(R)$ is a realizable one. Thus, all we have to show is that $\overline{\mathfrak{D}(A)} = \mathfrak{H}$ and $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$.

We will briefly repeat the framework of the proof of Theorem 9.

Let $C_{00}(E, (-\infty, +\infty)$ be the set of continuous compactly supported vector-valued functions $f(t)$ $(-\infty < t < +\infty)$ with values in a finite dimensional Hilbert space E. We introduce an inner product

$$(f, g) = \int\limits_{-\infty}^{+\infty} (G(dt)f(t), g(t))_E \tag{39}$$

for all $f, g \in C_{00}(E, (-\infty, +\infty))$. To construct a Hilbert space we identify with zero all the functions $f(t)$ such that $(f, f) = 0$, make a completion, and obtain a new Hilbert space $L^2_G(E)$.

Let $\mathfrak{D}_0$ be the set of the continuous vector-valued (with values in E) functions $f(t)$ such that not only

$$\int_{-\infty}^{+\infty} (dG(t)f(t), f(t))_E < \infty, \tag{40}$$

holds but also

$$\int_{-\infty}^{+\infty} t^2 (dG(t)f(t), f(t))_E < \infty, \tag{41}$$

is true. We introduce an operator $\hat{A}$ on $\mathfrak{D}_0$ in the following way

$$\hat{A}f(t) = tf(t). \tag{42}$$

Below we denote again by $\hat{A}$ the closure of Hermitian operator $\hat{A}$ (42). Moreover, $\hat{A}$ is self-adjoint in $L^2_G(E)$. Now let $\tilde{\mathfrak{H}}_+ = \mathfrak{D}(\hat{A})$ with an inner product

$$(f,g)_{\tilde{\mathfrak{H}}_+} = (f,g) + (\hat{A}f, \hat{A}g) \tag{43}$$

for all $f, g \in \tilde{\mathfrak{H}}_+$. We equip the space $L^2_G(E)$ with spaces $\tilde{\mathfrak{H}}_+$ and $\tilde{\mathfrak{H}}_-$:

$$\tilde{\mathfrak{H}}_+ \subset L^2_G(E) \subset \tilde{\mathfrak{H}}_-. \tag{44}$$

and denote by $\tilde{\mathcal{R}}$ the corresponding Riesz-Berezanskii operator, $\tilde{\mathcal{R}} \in [\tilde{\mathfrak{H}}_-, \tilde{\mathfrak{H}}_+]$. After straightforward calculations on the vectors $e(t) = e$, $e \in E$ we obtain

$$\tilde{\mathcal{R}}e = \frac{e}{1+t^2}, \quad e \in E. \tag{45}$$

Let us now consider the set

$$\mathfrak{D}(A) = \tilde{\mathfrak{H}}_+ \ominus \tilde{\mathcal{R}}E, \tag{46}$$

where by $\ominus$ we mean orthogonality in $\tilde{\mathfrak{H}}_+$. We define an operator A on $\mathfrak{D}(A)$ by the following expression

$$A = \hat{A}\Big|_{\mathfrak{D}(A)} \tag{47}$$

Obviously A is a closed Hermitian operator.

Since $V(z)$ is a member of the class $N_0(R)$ then (38) holds for all $e \in E$. Consequently, in the $(-)$-orthogonal decomposition

$$E = E_\infty \oplus F_\infty, \quad \text{where} \quad F_\infty = E_\infty^\perp$$

the first term $E_\infty = 0$. So that $E = F_\infty$ and (46) can be written as

$$\mathfrak{D}(A) = \tilde{\mathfrak{H}}_+ \ominus \tilde{\mathcal{R}}F_\infty.$$

Let us note again that in the formula above we are talking about $(+)$-orthogonal difference.

If we identify the space E with the space of functions $e(t) = e$, $e \in E$ we obtain

$$L^2_G(E) \ominus \overline{\mathfrak{D}(A)} = E_\infty. \tag{48}$$

The right-hand side of (48) is zero in our case and we can conclude that

$$\overline{\mathfrak{D}(A)} = L^2_G(E) = \mathfrak{H}.$$

Let us now show that $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$. We already found out that our operator A is densely defined. This implies that its defect subspaces coincide with the semi-defect subspaces. In particular, $\mathfrak{N}_{\pm i} = \mathfrak{N}'_{\pm i}$. Using the same technique that we used in the proof of Theorem 9 (see [7]) we obtain

$$\mathfrak{N}'_{\pm i} = \mathfrak{N}_{\pm i} = \left\{ f(t) \in L^2_G(E),\ f(t) = \frac{e}{t \pm i}, \quad e \in E \right\}. \tag{49}$$

For the pair of admissible operators $\Phi \in [\mathfrak{N}_i, \mathfrak{N}_{-i}]$ and $\Phi^* \in [\mathfrak{N}_{-i}, \mathfrak{N}_i]$ where

$$\Phi\left(\frac{e}{t-i}\right) = \frac{e}{t+i}, \quad e \in E. \tag{50}$$

we have that

$$\mathfrak{D}(T) = \mathfrak{D}(A) \dotplus (I - \Phi)\mathfrak{N}_i,$$
$$\mathfrak{D}(T^*) = \mathfrak{D}(A) \dotplus (I - \Phi^*)\mathfrak{N}_{-i}.$$

Direct calculations show that

$$(I - \Phi)\left(\frac{e}{t-i}\right) = \frac{e}{t-i} - \frac{e}{t+i} = \frac{2ie}{t^2+1}, \quad e \in E,$$

and

$$(I - \Phi^*)\left(\frac{e}{t+i}\right) = \frac{e}{t+i} - \frac{e}{t-i} = -\frac{2ie}{t^2+1}, \quad e \in E,$$

Therefore,

$$(I - \Phi)\mathfrak{N}_i = \left\{ \frac{2ie}{t^2+1}, \quad e \in E \right\}, \tag{51}$$

and

$$(I - \Phi^*)\mathfrak{N}_{-i} = \left\{ -\frac{2ie}{t^2+1}, \quad e \in E \right\}. \tag{52}$$

Applying Theorem 1 we conclude that $\mathfrak{D}(T) = \mathfrak{D}(T^*)$ if and only if $\mathfrak{N}_{\pm i} = 0$, which is not true. Therefore, the condition $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$ is satisfied and the proof of the theorem is complete.

Similar results for the class $N_1(R)$ can be obtained in the following two theorems.

Theorem 12. *Let θ be a l. s. c. d. s. of the form (28), $\dim E < \infty$ where A is a linear closed Hermitian O-operator and $\mathfrak{D}(T) = \mathfrak{D}(T^*)$. Then operator-valued function $V_\theta(\lambda)$ of the form (31), (32) belongs to the class $N_1(R)$.*

Proof. As in the Theorem 10 we already know that the operator-valued function $V_\theta(\lambda)$ belongs to the class $N(R)$. Therefore it is enough to show that

$$\int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty,$$

for all $e \in E$ and (37) holds.

Since it is given that A is closed Hermitian O-operator we can use Theorem 4 saying that for the system θ

$$\mathfrak{D}(T) = \mathfrak{D}(T^*) = \mathfrak{H}_+ = \mathfrak{D}(A^*).$$

This fact implies that the $(*)$-extension $\mathbb{A}$ coincides with operator T. Consequently, $\mathbb{A}^* = T^*$ and our system θ has a form

$$\theta = \begin{pmatrix} T & K & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & E \end{pmatrix}, \tag{53}$$

where

$$\operatorname{Im} T = \frac{T - T^*}{2i} = KJK^*.$$

Taking into account that $\dim E < \infty$ and $K : E \to \mathfrak{H}_-$ we conclude that $\dim \mathfrak{R}(\operatorname{Im} T) < \infty$.

Let

$$T = T_R + i\operatorname{Im} T,$$
$$T^* = T_R - i\operatorname{Im} T,$$

where

$$T_R = \frac{T + T^*}{2}.$$

In our case the operator K is acting from the space E into the space $\mathfrak{H}$. Therefore $Ke = g$ belongs to $\mathfrak{H}$ for all $e \in E$. For the operator-valued function $V_\theta(\lambda)$ we can derive an integral representation for all $f \in E$

$$\begin{aligned} \big(V_\theta(\lambda)f, f\big)_E &= \big(K^*(T_R - \lambda I)^{-1}Kf, f\big)_E = \left(K^* \int_{-\infty}^{+\infty} \frac{dE(t)}{t - \lambda} Kf, f \right)_E \\ &= \int_{-\infty}^{+\infty} \frac{d\big(K^*E(t)Kf, f\big)_E}{t - \lambda}, \end{aligned} \tag{54}$$

where $E(t)$ is the complete set of spectral orthoprojections of the operator T_R. Denote

$$G(t) = K^*E(t)K.$$

Then

$$\int_{-\infty}^{+\infty} d(G(t)e,e) = \int_{-\infty}^{+\infty} d(K^*E(t)Ke,e) = \int_{-\infty}^{+\infty} d(E(t)Ke,Ke)$$
$$= \int_{-\infty}^{+\infty} d(E(t)g,g) = (g,g)\int_{-\infty}^{+\infty} dE(t) = (g,g)$$
$$= (Ke,Ke) = (K^*Ke,e) = (\operatorname{Im} Te,e) < \infty,$$

for all $e \in E$. Using standard techniques we obtain the representation (37) from the representation (54). This completes the proof of the theorem.

Theorem 13. *Suppose that an operator-valued function $V(z)$ is acting on a finite-dimensional Hilbert space E and belongs to the class $N_1(R)$. Then it admits a realization by the system θ of the form (28) with a preassigned directional operator J for which $I+iV(-i)J$ is invertible, a linear closed regular Hermitian O-operator A with a non-dense domain, and $\mathfrak{D}(T) = \mathfrak{D}(T^*)$.*

Proof. Similarly to Theorem 11 we can say that since $N_1(R)$ is a subclass of $N(R)$ then it is sufficient to show that operator A is a closed Hermitian O-operator with a non-dense domain and $\mathfrak{D}(T) = \mathfrak{D}(T^*)$.

Once again we introduce an operator $\hat{A}$ by the formula (42), an operator A by the formula (47) and note that

$$\mathfrak{D}(A) = \tilde{\mathfrak{H}}_+ \ominus \tilde{\mathcal{R}}E.$$

Let us recall, that since $V(z)$ belongs to the class $N_1(R)$ then

$$\int_{-\infty}^{+\infty} (dG(t)e,e)_E < \infty, \quad \forall e \in E.$$

That means that in the $(-)$-orthogonal decomposition

$$E = E_\infty \oplus F_\infty$$

the second term $F_\infty = 0$ and therefore $E = E_\infty$. Then

$$\mathfrak{D}(A) = \tilde{\mathfrak{H}}_+ \ominus \tilde{\mathcal{R}}E_\infty.$$

Combining this, formula (48), and the fact that $E_\infty \neq 0$ we obtain that $\overline{\mathfrak{D}(A)} \neq \mathfrak{H} = L^2_G(E)$. Relying on the proof of Theorem 9 (see [7]) we let

$$A_1 = \hat{A}\Big|_{\mathfrak{D}(A_1)}, \qquad \mathfrak{D}(A_1) = \tilde{\mathfrak{H}}_+ \ominus \tilde{\mathcal{R}}E_\infty.$$

The following obvious inclusions hold: $A \subset A_1 \subset \hat{A}$. Moreover, a set

$$\mathfrak{D}(A_1) = \tilde{\mathfrak{H}}_+ \ominus \tilde{\mathcal{R}} E_\infty$$

in our case coincides with $\mathfrak{D}(A)$ and operator A_1 (defined on $\mathfrak{D}(A_1)$) with A. Now it is not difficult to see that

$$\mathfrak{D}(A^*) = \mathfrak{H}_+ = \tilde{\mathfrak{H}}_+,$$

the rigged Hilbert space $\tilde{\mathfrak{H}}_+ \subset \mathfrak{H} \subset \tilde{\mathfrak{H}}_-$ coincides with $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$ and $\mathcal{R} = \tilde{\mathcal{R}}$. Indeed, $\tilde{\mathfrak{H}}_+ = \mathfrak{D}(\hat{A})$ by the definition, in [7] we have shown that $\mathfrak{D}(A_1^*) = \mathfrak{D}(\hat{A})$, and $D(A_1) = D(A)$ above. All together it yields $\mathfrak{H}_+ = \tilde{\mathfrak{H}}_+$.

Let $\mathfrak{N}'_{\pm i}$ be the semidefect subspaces of operator A and $\mathfrak{N}^0_{\pm i}$ be the defect subspaces of operator A_1, described in the second part of the proof of Theorem 9 (see [7]). It was shown that

$$\mathfrak{N}^0_{\pm i} = \left\{ f(t) \in L^2_G(E),\ f(t) = \frac{e}{t \pm i},\ \ e \in E_\infty \right\}, \tag{55}$$

and

$$\mathfrak{N}'_{\pm i} = \mathfrak{N}_i \ominus \mathfrak{N}^0_{\pm i}, \tag{56}$$

where $\mathfrak{N}_i$ are defect spaces of the operator A. In our case $A = A_1$ therefore

$$\mathfrak{N}'_{\pm i} = 0.$$

This implies that the semidefect numbers of operator A are equal to zero. Hence, A is an O-operator.

Note that A is also a regular Hermitian operator. Thus, Theorem 4 is applicable and yields

$$\mathfrak{D}(T) = \mathfrak{D}(T^*).$$

This completes the proof of the theorem.

The following two theorems will complete our framework by establishing direct and inverse realization results for the remaining subclass of realizable operator-valued R-functions $N_{01}(R)$.

Theorem 14. *Let θ be a l. s. c. d. s. of the form* (28), $\dim E < \infty$ *where A is a linear closed Hermitian operator with non-dense domain and $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$. Then the operator-valued function $V_\theta(z)$ of the form* (31), (32) *belongs to the class $N_{01}(R)$.*

Proof. We know that $V_\theta(z)$ belongs to the class $N(R)$. To prove the statement of the theorem we only have to show that in the $(-)$-orthogonal decomposition $E = E_\infty \oplus F_\infty$

both components are non-zero. In other words we have to show the existence of such vectors $e \in E$ that

$$\int_{-\infty}^{+\infty} d(G(t)e, e) = \infty, \tag{57}$$

and vectors $f \in E$, $f \neq 0$ that

$$\int_{-\infty}^{+\infty} d(G(t)f, f) < \infty. \tag{58}$$

Let $\mathfrak{H}_0 = \overline{\mathfrak{D}(A)}$ and $\mathfrak{L} = \mathfrak{H} \ominus \mathfrak{H}_0$. Since $\overline{\mathfrak{D}(A)} = \mathfrak{H}_0 \neq \mathfrak{H}$, $\mathfrak{L}$ is non-empty. $K^{-1}\mathfrak{L}$ is obviously a subset of E. Moreover, according to Theorem 7 for all $f \in K^{-1}\mathfrak{L}$ (58) holds. Thus, $K^{-1}\mathfrak{L}$ is a non-zero subset of E_∞.

Now we have to show that the vectors satisfying (57) make a non-zero subset of E as well. Indeed, the condition

$$\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$$

implies that a certain part of $\mathfrak{R}(K) \subseteq \overline{\mathfrak{R}(\mathbb{A} - \mathbb{A}^*) + \mathfrak{L}} \subseteq L \dotplus \mathfrak{L}$ where L was defined in Theorem 7 essentially belongs to L. Otherwise we could have re-traced our steps and show that $\mathfrak{D}(T) = \mathfrak{D}(T^*)$. Therefore, there exist $g \in \mathfrak{H}_-$, $g \notin \mathfrak{L}$, $f \in E$ such that $Kf = g \notin \mathfrak{L}$. Then according to Theorem 7 for this $f \in E$ (57) holds. The proof of the theorem is complete.

Theorem 15. *Suppose that an operator-valued function $V(z)$ is acting on a finite-dimensional Hilbert space E and belongs to the class $N_{01}(R)$. Then it admits a realization by the system θ of the form* (28) *with a preassigned directional operator J for which $I + iV(-i)J$ is invertible, a linear closed regular Hermitian operator A with a non-dense domain, and $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$.*

Proof. Once again all we have to show is that $\overline{\mathfrak{D}(A)} \neq \mathfrak{H}$. We have already mentioned (48) that $L^2_G(E) \ominus \overline{\mathfrak{D}(A)} = E_\infty$. This implies that $\mathfrak{D}(A)$ is dense in $\mathfrak{H}$ if and only if $E_\infty = 0$. Since the class $N_{01}(R)$ assumes the existence of non-zero vectors $f \in E$ such that (58) is true we can conclude that $E_\infty \neq 0$ and therefore $\overline{\mathfrak{D}(A)} \neq \mathfrak{H}$.

In the proofs of Theorems 11 and 13 we have shown that $\mathfrak{D}(T) = \mathfrak{D}(T^*)$ in case when $F_\infty = 0$. If $F_\infty \neq 0$ then $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$. The definition of the class $N_{01}(R)$ implies that $F_\infty \neq 0$. Thus we have $\mathfrak{D}(T) \neq \mathfrak{D}(T^*)$. The proof is complete.

Let us consider examples of the realization in the classes $N(R)$.

Example 1. This example is to illustrate the realization in $N_0(R)$ class. Let

$$Tx = \frac{1}{i}\frac{dx}{dt},$$

with

$$\mathfrak{D}(T) = \left\{x(t) \,\Big|\, x(t) - \text{ abs. continuous}, x'(t) \in L^2_{[0,l]}, x(0) = 0\right\},$$

be differential operator in $\mathfrak{H} = L^2_{[0,l]}$ $(l > 0)$. Obviously,

$$T^*x = \frac{1}{i}\frac{dx}{dt},$$

with

$$\mathfrak{D}(T^*) = \left\{x(t) \,\Big|\, x(t) - \text{ abs. continuous}, x'(t) \in L^2_{[0,l]}, x(l) = 0\right\},$$

is its adjoint. Consider a Hermitian operator A [1]

$$\begin{aligned} Ax &= \frac{1}{i}\frac{dx}{dt}, \\ \mathfrak{D}(A) &= \left\{x(t) \,\Big|\, x(t) - \text{ abs. continuous}, x'(t) \in L^2_{[0,l]}, x(0) = x(l) = 0\right\}, \end{aligned}$$

and its adjoint A^*

$$\begin{aligned} A^*x &= \frac{1}{i}\frac{dx}{dt}, \\ \mathfrak{D}(A^*) &= \left\{x(t) \,\Big|\, x(t) - \text{ abs. continuous}, x'(t) \in L^2_{[0,l]}\right\}. \end{aligned}$$

Then $\mathfrak{H}_+ = \mathfrak{D}(A^*) = W^1_2$ is a Sobolev space with scalar product

$$(x, y)_+ = \int_0^l x(t)\overline{y(t)}\,dt + \int_0^l x'(t)\overline{y'(t)}\,dt.$$

Construct rigged Hilbert space [9]

$$W^1_2 \subset L^2_{[0,l]} \subset (W^1_2)_-,$$

and consider operators

$$\begin{aligned} \mathbb{A}x &= \frac{1}{i}\frac{dx}{dt} + ix(0)\left[\delta(x-l) - \delta(x)\right], \\ \mathbb{A}^*x &= \frac{1}{i}\frac{dx}{dt} + ix(l)\left[\delta(x-l) - \delta(x)\right], \end{aligned}$$

where $x(t) \in W^1_2$, $\delta(x)$, $\delta(x-l)$ are delta-functions in $(W^1_2)_-$. It is easy to see that

$$\mathbb{A} \supset T \supset A, \qquad \mathbb{A}^* \supset T^* \supset A,$$

and

$$\theta = \begin{pmatrix} \frac{1}{i}\frac{dx}{dt} + ix(0)[\delta(x-l) - \delta(x)] & K & -1 \\ & & \\ W_1^2 \subset L^2_{[0,l]} \subset (W_2^1)_- & & \mathbb{C}^1 \end{pmatrix} \qquad (J = -1)$$

is the Brodskiĭ-Livšic rigged operator colligation where

$$Kc = c \cdot \frac{1}{\sqrt{2}}[\delta(x-l) - \delta(x)], \quad (c \in \mathbb{C}^1)$$
$$K^* x = \left(x, \frac{1}{\sqrt{2}}[\delta(x-l) - \delta(x)]\right) = \frac{1}{\sqrt{2}}[x(l) - x(0)],$$

and $x(t) \in W_2^1$. Also

$$\frac{\mathbb{A} - \mathbb{A}^*}{2i} = -\left(\cdot, \frac{1}{\sqrt{2}}[\delta(x-l) - \delta(x)]\right) \frac{1}{\sqrt{2}}[\delta(x-l) - \delta(x)].$$

The characteristic function of this colligation can be found as follows

$$W_\theta(\lambda) = I - 2iK^*(\mathbb{A} - \lambda I)^{-1} KJ = e^{i\lambda l}.$$

Consider the following R-function (hyperbolic tangent)

$$V(\lambda) = -i \tanh\left(\frac{i}{2}\lambda l\right).$$

Obviously this fucntion can be realized as follows

$$V(\lambda) = -i \tanh\left(\frac{i}{2}\lambda l\right) = -i\frac{e^{\frac{i}{2}\lambda l} - e^{-\frac{i}{2}\lambda l}}{e^{\frac{i}{2}\lambda l} + e^{-\frac{i}{2}\lambda l}} = -i\frac{e^{i\lambda l} - 1}{e^{i\lambda l} + 1}$$
$$= i\left[W_\theta(\lambda) + I\right]^{-1}\left[W_\theta(\lambda) - I\right] J. \qquad (J = -1)$$

The following simple example showing the realization for $N_1(R)$ class.

Example 2. Consider bounded linear operator in $\mathbb{C}^2$:

$$T = \begin{pmatrix} i & i \\ -i & 1 \end{pmatrix}$$

Let x be an element of $\mathbb{C}^2$ such that

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and φ be a row vector $\varphi = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and let $J = 1$. Obviously,

$$T^* = \begin{pmatrix} -i & i \\ -i & 1 \end{pmatrix}.$$

It is clear that $\mathfrak{D}(T) = \mathfrak{D}(T^*)$. Now we can find

$$\frac{T - T^*}{2i} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

and show that φ above is the only channel vector such that

$$\frac{T - T^*}{2i} x = (x, \varphi) J \varphi.$$

Thus, operator T can be included in the system

$$\theta = \begin{pmatrix} T & K & J \\ \mathbb{C}^2 & & \mathbb{C}^1 \end{pmatrix},$$

with

$$Kc = \begin{pmatrix} c & 0 \end{pmatrix}, \quad c \in \mathbb{C}^1$$

$$K^* x = x_1, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{C}^2,$$

Then $W_\theta(\lambda)$ is represented by the formula

$$W_\theta(\lambda) = \frac{\lambda^2 + (1 - i)\lambda - 1 - 1}{\lambda^2 - (1 + i)\lambda - 1 + i}.$$

Its linear-fractional transformation is a R-function and

$$V_\theta(\lambda) = \frac{1 - \lambda}{\lambda^2 - \lambda - 1}$$

can therefore be realized as follows

$$V_\theta(\lambda) = i \left[W_\theta(\lambda) + I \right]^{-1} \left[W_\theta(\lambda) - I \right] J.$$

Example 3. In order to present the realization in $N_{01}(R)$ class we will use Examples 1 and 2.

Consider the system

$$\theta = \begin{pmatrix} \mathbb{A} & K & J \\ W_1^2 \otimes \mathbb{C}^2 \subset L^2_{[0,l]} \otimes \mathbb{C}^2 \subset (W_2^1)_- \otimes \mathbb{C}^2 & & \mathbb{C}^2 \end{pmatrix},$$

where $\mathbb{A}$ is a diagonal block-matrix

$$\mathbb{A} = \begin{pmatrix} \mathbb{A}_1 & 0 \\ 0 & T \end{pmatrix},$$

with

$$\mathbb{A}_1 = \frac{1}{i}\frac{dx}{dt} + ix(0)\left[\delta(x-l) - \delta(x)\right]$$

from Example 1, and

$$T = \begin{pmatrix} i & i \\ -i & 1 \end{pmatrix},$$

from Example 2. Operator K here is defined as a diagonal operator block-matrix

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix},$$

with operators K_1 and K_2 from Examples 1 and 2, respectively,

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It can be easily shown that

$$W_\theta(\lambda) = \begin{pmatrix} e^{i\lambda l} & 0 \\ 0 & \dfrac{\lambda^2 + (1-i)\lambda - 1 - 1}{\lambda^2 - (1+i)\lambda - 1 + i} \end{pmatrix},$$

and

$$V_\theta(\lambda) = \begin{pmatrix} -i\tanh\left(\frac{i}{2}\lambda l\right) & 0 \\ 0 & \dfrac{1-\lambda}{\lambda^2 - \lambda - 1} \end{pmatrix},$$

is an operator-valued function of class $N_{01}(R)$.

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