REALIZATION THEOREMS FOR OPERATOR-VALUED R-FUNCTIONS

S.V. BELYI AND E.R. TSEKANOVSKII

Dedicated to the memory of Professor Israel Glazman

In this paper we consider realization problems for operator-valued $R$-functions acting on a Hilbert space $E$ ($\dim E < \infty$) as linear-fractional transformations of the transfer operator-valued functions (characteristic functions) of linear stationary conservative dynamic systems (Brodskiĭ-Livšic rigged operator colligations). We give complete proofs of both the direct and inverse realization theorems announced in [6], [7].

1. Introduction

Realization theory of different classes of operator-valued (matrix-valued) functions as transfer operator-functions of linear systems plays an important role in modern operator and systems theory. Almost all realizations in the modern theory of non-selfadjoint operators and its applications deal with systems (operator colligations) in which the main operators are bounded linear operators [8], [10-14], [17], [21]. The realization with an unbounded operator as a main operator in a corresponding system has not been investigated thoroughly because of a number of essential difficulties usually related to unbounded non-selfadjoint operators.

We consider realization problems for operator-valued $R$-functions acting on a finite dimensional Hilbert space $E$ as linear-fractional transformations of the transfer operator-functions of linear stationary conservative dynamic systems (l.s.c.d.s.) $\theta$ of the form

\[
\begin{cases}
(\mathbb{A} - zI)x = KJ\varphi_-
\quad (\text{Im } \mathbb{A} = KJK^*), \\
\varphi_+ = \varphi_- - 2iK^* x
\end{cases}
\]
or

\[ \theta = \begin{pmatrix} A & J \\ H & E \end{pmatrix}. \]

In the system \( \theta \) above \( A \) is a bounded linear operator, acting from \( H^+ \subset H \subset H^- \). \( A \) is a Hermitian operator in \( H \), \( T \) is a non-Hermitian operator in \( H \), \( K \) is a linear bounded operator from \( E \) into \( H^- \), \( J = J^* = J^{-1} \), \( \varphi_ \pm \in E \), \( \varphi_- \) is an input vector, \( \varphi_+ \) is an output vector, and \( x \in H^+ \) is a vector of the inner state of the system \( \theta \). The operator-valued function

\[ W_\theta(z) = I - 2iK^*(A - zI)^{-1}KJ \quad (\varphi_+ = W_\theta(z)\varphi_-), \]

is the transfer operator-function of the system \( \theta \).

We establish criteria for a given operator-valued \( R \)-function \( V(z) \) to be realized in the form

\[ V(z) = i[W_\theta(z) + I]^{-1}[W_\theta(z) - I]J. \]

It is shown that an operator-valued \( R \)-function

\[ V(z) = Q + F \cdot z + \int_{-\infty}^{+\infty} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) dG(t), \]

acting on a Hilbert space \( E \) (dim \( E < \infty \)) with some invertibility condition can be realized if and only if

\[ F = 0 \quad \text{and} \quad Qe = \int_{-\infty}^{+\infty} \frac{t}{1 + t^2} dG(t)e, \]

for all \( e \in E \) such that

\[ \int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty. \]

Moreover, if two realizable operator-valued \( R \)-functions are different only by a constant term then they can be realized by two systems \( \theta_1 \) and \( \theta_2 \) with corresponding non-selfadjoint operators that have the same Hermitian part \( A \).

The rigged operator colligation \( \theta \) mentioned above is exactly an unbounded version of the well known Brodskii-Livšic bounded operator colligation \( \alpha \) of the form [11]

\[ \alpha = \begin{pmatrix} T \phantom{J} K \\ H \phantom{E} J \\ E \end{pmatrix} \quad (\text{Im } T = KJK^*), \]

with a bounded linear operator \( T \) in \( H \) (and without rigged Hilbert spaces).

To prove the direct and inverse realization theorems for operator-valued \( R \)-functions we build a functional model which generally speaking is an unbounded version of the
Brodskii-Livšic model with diagonal real part. This model for bounded linear operators was constructed in [11].

When this paper was submitted for publication, an article by D. Arov and M. Nudelman [5] appeared considering realization problem for another class of operator-valued functions (contractive) but not in terms of rigged operator colligations. At the end of this paper there is an example showing how a given \( R \)-function can be realized by a rigged operator colligation.

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2. Preliminaries

In this section we recall some basic definitions and results that will be used in the proof of the realization theorem.

The Rigged Hilbert Spaces. Let \( \mathfrak{H} \) denote a Hilbert space with inner product \((x, y)\) and let \( A \) be a closed linear Hermitian operator, i.e. \((Ax, y) = (x, Ay)\) (\(\forall x, y \in \mathcal{D}(A)\)), acting in the Hilbert space \( \mathfrak{H} \) with generally speaking, non-dense domain \( \mathcal{D}(A) \). Let \( \mathfrak{H}_0 = \overline{\mathcal{D}(A)} \) and \( A^* \) be the adjoint to the operator \( A \) (we consider \( A \) acting from \( \mathfrak{H}_0 \) into \( \mathfrak{H} \)).

Now we are going to equip \( \mathfrak{H} \) with spaces \( \mathfrak{H}^+ \) and \( \mathfrak{H}^- \) called, respectively, spaces with positive and negative norms [9]. We denote \( \mathfrak{H}^+ = \mathcal{D}(A^*) \) (\(\overline{\mathcal{D}(A^*)} = \mathfrak{H} \)) with inner product \((1) (f, g)^+ = (f, g) + (A^*f, A^*g) \quad (f, g \in \mathfrak{H}^+)\),

and then construct the rigged Hilbert space \( \mathfrak{H}^+ \subset \mathfrak{H} \subset \mathfrak{H}^- \). Here \( \mathfrak{H}^- \) is the space of all linear functionals over \( \mathfrak{H}^+ \) that are continuous with respect to \( \| \cdot \|^+ \). The norms of these spaces are connected by the relations \( \|x\| \leq \|x\|^+ \quad (x \in \mathfrak{H}^+) \), and \( \|x\|^- \leq \|x\| \quad (x \in \mathfrak{H}) \). It is well known that there exists an isometric operator \( \mathcal{R} \) which maps \( \mathfrak{H}^- \) onto \( \mathfrak{H}^+ \) such that

\[(2) (x, y)_- = (x, \mathcal{R}y) = (\mathcal{R}x, y) = (\mathcal{R}x, \mathcal{R}y)_+ \quad (x, y \in \mathfrak{H}^-), \]
\[(u, v)_+ = (u, \mathcal{R}^{-1}v) = (\mathcal{R}^{-1}u, v) = (\mathcal{R}^{-1}u, \mathcal{R}^{-1}v)_- \quad (u, v \in \mathfrak{H}^+).\]

The operator \( \mathcal{R} \) will be called the Riesz-Berezanskii operator. In what follows we use symbols \((+), (-)\), and \((-)\) to indicate the norms \(\| \cdot \|^+, \| \cdot \|, \) and \(\| \cdot \|^-\) by which geometrical and topological concepts are defined in \( \mathfrak{H}^+, \mathfrak{H}, \) and \( \mathfrak{H}^- \).

Analogues of von Neumann's formulae. It is easy to see that for a Hermitian operator \( A \) in the above settings \( \mathcal{D}(A) \subset \mathcal{D}(A^*) (= \mathfrak{H}^+) \) and \( A^*y = PAy \quad (\forall y \in \mathcal{D}(A)) \), where \( P \) is an orthogonal projection of \( \mathfrak{H} \) onto \( \mathfrak{H}_0 \). We put

\[(3) \mathcal{L} := \mathfrak{H} \ominus \mathfrak{H}_0 \quad \mathfrak{M}_\lambda := (A - \lambda I) \mathcal{D}(A) \quad \mathfrak{M}_\lambda := (\mathfrak{M}_\lambda)^\perp \]
The subspace $\mathcal{N}_\lambda$ is called a \textit{defect subspace} of $A$ for the point $\bar{\lambda}$. The cardinal number $\dim \mathcal{N}_\lambda$ remains constant when $\lambda$ is in the upper half-plane. Similarly, the number $\dim \mathcal{N}_\lambda$ remains constant when $\lambda$ is in the lower half-plane. The numbers $\dim \mathcal{N}_\lambda$ and $\dim \mathcal{N}_{\bar{\lambda}}$ ($\text{Im} \lambda < 0$) are called the \textit{defect numbers} or \textit{deficiency indices} of operator $A$ [1]. The subspace $\mathcal{N}_\lambda$ which lies in $\mathcal{H}_+$ is the set of solutions of the equation $A^* g = \lambda P g$.

Let now $P_\lambda$ be the orthogonal projection onto $\mathcal{N}_\lambda$, set

$$B_\lambda = P_\lambda L,$$

$$\mathcal{N}_\lambda' = \mathcal{N}_\lambda \ominus \overline{B_\lambda}$$

It is easy to see that $\mathcal{N}_\lambda' = \mathcal{N}_\lambda \cap \mathcal{H}_0$ and $\mathcal{N}_\lambda'$ is the set of solutions of the equation $A^* g = \lambda g$ (see [25]), when $A^* : \mathcal{H} \to \mathcal{H}_0$ is the adjoint operator to $A$.

The subspace $\mathcal{N}_\lambda'$ is the defect subspace of the densely defined Hermitian operator $P A$ on $\mathcal{H}_0$ ([22]). The numbers $\dim \mathcal{N}_\lambda'$ and $\dim \mathcal{N}_{\bar{\lambda}}'$ ($\text{Im} \lambda < 0$) are called \textit{semi-defect numbers} or the \textit{semi-deficiency indices} of the operator $A$ [16]. The von Neumann formula

$$\mathcal{H}_+ = \mathcal{D}(A^*) = \mathcal{D}(A) + \mathcal{N}_\lambda + \mathcal{N}_{\bar{\lambda}}, \quad (\text{Im} \lambda \neq 0),$$

holds, but this decomposition is not direct for a non-densely defined operator $A$. There exists a generalization of von Neumann’s formula [3], [24] to the case of a non-densely defined Hermitian operator (direct decomposition).

We call an operator $A$ \textit{regular}, if $P A$ is a closed operator in $\mathcal{H}_0$. For a regular operator $A$ we have

$$\mathcal{H}_+ = \mathcal{D}(A) + \mathcal{N}_\lambda' + \mathcal{N}_{\bar{\lambda}}', \quad (\text{Im} \lambda \neq 0)$$

where $\mathcal{N} := \mathcal{R} L$. This is a generalization of von Neumann’s formula. For $\lambda = \pm i$ we obtain the $(+)$-orthogonal decomposition

$$\mathcal{H}_+ = \mathcal{D}(A) \oplus \mathcal{N}_i \oplus \mathcal{N}_{-i} \oplus \mathcal{N}.$$  

Let $\tilde{A}$ be a closed Hermitian extension of the operator $A$. Then $\mathcal{D}(\tilde{A}) \subset \mathcal{H}_+$ and $P \tilde{A} x = A^* x \ (\forall x \in \mathcal{D}(\tilde{A}))$. According to [25] a closed Hermitian extension $\tilde{A}$ is said to be \textit{regular} if $\mathcal{D}(\tilde{A})$ is $(+)$-closed. According to the theory of extensions of closed Hermitian operators $A$ with non-dense domain [16], an operator $U \ (\mathcal{D}(U) \subset \mathcal{N}_i, \mathcal{R}(U) \subset \mathcal{N}_{-i})$ is called an \textit{admissible operator} if $(U - I) f_i \in \mathcal{D}(A) \ (f_i \in \mathcal{D}(U))$ only for $f_i = 0$. Then (see [4]) any symmetric extension $\tilde{A}$ of the non-densely defined closed Hermitian operator $A$, is defined by an isometric admissible operator $U$, $\mathcal{D}(U) \subset \mathcal{N}_i, \mathcal{R}(U) \subset \mathcal{N}_{-i}$ by the formula

$$\tilde{A} f_A = A f_A + (-i f_i - iU f_i), \quad f_A \in \mathcal{D}(A)$$
where $\mathcal{D}(A) = \mathcal{D}(A) + (U-I)\mathcal{D}(U)$. The operator $\tilde{A}$ is self-adjoint if and only if $\mathcal{D}(U) = \mathcal{N}$ and $\mathcal{R}(U) = \mathcal{N}_{-1}$.

Let us denote now by $P_{\mathcal{N}}^+ \mathcal{H}$ the orthogonal projection operator in $\mathcal{H}$ onto $\mathcal{N}$. We introduce a new inner product $(\cdot, \cdot)_1$ defined by

$$(9) \quad (f, g)_1 = (f, g)_+ + (P_{\mathcal{N}}^+ f, P_{\mathcal{N}}^+ g)_+$$

for all $f, g \in \mathcal{H}$. The obvious inequality

$$\|f\|_2^2 \leq \|f\|_1^2 \leq 2\|f\|_2^2$$

shows that the norms $\|\cdot\|_+$ and $\|\cdot\|_1$ are topologically equivalent. It is easy to see that the spaces $\mathcal{D}(A), \mathcal{N}_+, \mathcal{N}_-, \mathcal{N}$ are (1)-orthogonal. We write $\mathcal{M}_1$ for the Hilbert space $\mathcal{M} = \mathcal{N}_+ \oplus \mathcal{N}_- \oplus \mathcal{N}$ with inner product $(f, g)_1$. We denote by $\mathcal{H}_{+1}$ the space $\mathcal{H}_+$ with norm $\|\cdot\|_1$, and by $\mathcal{R}_1$ the corresponding Riesz-Berezanskii operator related to the rigged Hilbert space $\mathcal{H}_{+1} \subset \mathcal{H} \subset \mathcal{H}_{-1}$. The following theorem gives a characterization of the regular extensions for a regular closed Hermitian operator $A$ (see [4]).

**Theorem 1.** I. For each closed Hermitian extension $\tilde{A}$ of a regular operator $A$ there exists a (1)-isometric operator $V = V(\tilde{A})$ on $\mathcal{M}_1$ with the properties: a) $\mathcal{D}(V)$ is (+)-closed and belongs to $\mathcal{N} \oplus \mathcal{N}_+$, $\mathcal{R}(V) \subset \mathcal{N} \oplus \mathcal{N}_-$; b) $Vh = h$ only for $h = 0$, and $\mathcal{D}(\tilde{A}) = \mathcal{D}(A) \oplus (I + V)\mathcal{D}(V)$.

Conversely, for each (1)-isometric operator $V$ with the properties a) and b) there exists a closed Hermitian extension $\tilde{A}$ in the sense indicated.

II. The extension $\tilde{A}$ is regular if and only if the manifold $\mathcal{R}(I + V)$ is (1)-closed.

III. The operator $\tilde{A}$ is self-adjoint if and only if $\mathcal{D}(V) = \mathcal{N} \oplus \mathcal{N}_+$, $\mathcal{R}(V) = \mathcal{N} \oplus \mathcal{N}_-$. The following theorem can be found in [16].

**Theorem 2.** Let $\tilde{A}$ be a regular self-adjoint extension of a regular Hermitian operator $A$, that is determined by an admissible operator $U$ and let

$$(10) \quad \hat{\mathcal{N}}_i = \left\{ f_i \in \mathcal{N}_i, (U - I)f_i \in \mathcal{H}_0 \right\}.$$  

Then

$$(11) \quad \mathcal{H}_+ = \mathcal{D}(\tilde{A}) + (U + I)\hat{\mathcal{N}}_i.$$  

**Bi-extensions.** Denote by $[\mathcal{H}_1, \mathcal{H}_2]$ the set of all linear bounded operators acting from the Hilbert space $\mathcal{H}_1$ into the Hilbert space $\mathcal{H}_2$.  

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**Definition.** An operator $A \in [\mathcal{H}_+, \mathcal{H}_-]$ is a *bi-extension* of $A$ if both $A \supset A$ and $A^* \supset A$.

If $A = A^*$, then $A$ is called a self-adjoint bi-extension of the operator $A$. We write $\mathfrak{S}(A)$ for the class of bi-extensions of $A$. This class is closed in the weak topology and is invariant under taking adjoints. The following theorem from [4], [25] gives a description of $\mathfrak{S}(A)$.

**Theorem 3.** Every bi-extension $A$ of a regular Hermitian operator $A$ has the form:

$$A = \mathcal{P}_\mathfrak{D}(A) + [A^* + \mathcal{R}_1^{-1}(Q - \frac{i}{2}P^{+}_{\mathfrak{M}} + \frac{i}{2}P^{-}_{\mathfrak{N}})]P^{+}_{\mathfrak{M}}$$

(12) $$A^* = \mathcal{P}_\mathfrak{D}(A) + [A^* + \mathcal{R}_1^{-1}(Q^* - \frac{i}{2}P^{+}_{\mathfrak{M}} + \frac{i}{2}P^{-}_{\mathfrak{N}})]P^{+}_{\mathfrak{M}}$$

(13)

where $Q$ is an arbitrary operator in $[\mathfrak{M}, \mathfrak{M}]$ and $Q^*$ is its adjoint with respect to the $(1)$-metric.

**Corollary 1.** Every self-adjoint bi-extension $A$ of the regular Hermitian operator $A$ is of the form:

$$A = \mathcal{P}_\mathfrak{D}(A) + [A^* + \mathcal{R}_1^{-1}(S - \frac{i}{2}P^{+}_{\mathfrak{M}} + \frac{i}{2}P^{-}_{\mathfrak{N}})]P^{+}_{\mathfrak{M}},$$

(14)

where $S$ is an arbitrary $(1)$-self-adjoint operator in $[\mathfrak{M}, \mathfrak{M}]$.

Let $A$ be a bi-extension of a Hermitian operator $A$. The operator $\hat{A}f = Af$, $\mathfrak{D}(\hat{A}) = \{f \in \mathfrak{H}, Af \in \mathfrak{H}\}$ is called the *quasi-kernel* of $A$. If $A = A^*$ and $\hat{A}$ is a quasi-kernel of $A$ such that $A \neq \hat{A}$, $\hat{A}^* = \hat{A}$ then $A$ is said to be a *strong* self-adjoint bi-extension of $A$.

**Classes $\Omega_A$ and $\Lambda_A$. ($\ast$)-extensions.** Let $A$ be a closed Hermitian operator.

**Definition.** We say that a closed densely defined linear operator $T$ acting on the Hilbert space $\mathfrak{H}$ belongs to the class $\Omega_A$ if:

1. $T \supset A$ and $T^* \supset A$;
2. $(-i)$ is a regular point of $T$.

It was mentioned in [4] that sets $\mathfrak{D}(T)$ and $\mathfrak{D}(T^*)$ are (+)-closed, the operators $T$ and $T^*$ are (+, ·)-bounded. The following theorem [25] is an analogue of von Neumann's formulae for the class $\Omega_A$.

Footnote: The condition, that $(-i)$ is a regular point in the definition of the class $\Omega_A$ is not essential. It is sufficient to require the existence of some regular point for $T$. 

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Theorem 4. I. To each operator of the class $\Omega_A$ there corresponds an operator $M$ on the space $\mathcal{M}_1$ with the following properties:

1. $\mathcal{D}(M) = \mathcal{N}_i' \oplus \mathcal{N}$, and $\mathcal{R}(M) = \mathcal{N}_i' \oplus \mathcal{N}$;
2. $Mx + x = 0$ only for $x = 0$, and $M^*x + x = 0$ only for $x = 0$. Moreover, the following hold:

\[ \mathcal{D}(T) = \mathcal{D}(A) \oplus (M + I)(\mathcal{N}_i' \oplus \mathcal{N}), \]

\[ \mathcal{D}(T^*) = \mathcal{D}(A) \oplus (M^* + I)(\mathcal{N}_{-i} \oplus \mathcal{N}). \]

II. Conversely, for each pair of (1)-adjoint operators $M$ and $M^*$ in $[\mathcal{M}_1, \mathcal{M}_1]$ satisfying (1) and (2) above, formulas (15) and (16) give a corresponding operator $T$ in the class $\Omega_A$. Moreover, if $f = g + (M + I)\varphi$, $g \in \mathcal{D}(A)$, and $\varphi \in \mathcal{N}_i' \oplus \mathcal{N}$ then

\[ Tf = Ag + A^*(I + M)\varphi + i\mathcal{R}_1^{-1}P_{\mathcal{M}}^+(I - M)\varphi \quad (f \in \mathcal{D}(T)). \]

Similarly, if $f = g + (M^* + I)\psi$, $g \in \mathcal{D}(A)$, and $\psi \in \mathcal{N}_{-i} \oplus \mathcal{N}$, then

\[ T^*f = Ag + A^*(I + M^*)\psi + i\mathcal{R}_1^{-1}P_{\mathcal{M}}^+(M^* - I)\psi \quad (f \in \mathcal{D}(T)). \]

Definition. An operator $\mathcal{A}$ in $[\mathcal{N}_+, \mathcal{N}_-]$ is called a ($*$)-extension of an operator $T$ from the class $\Omega_A$ if both $\mathcal{A} \supset T$ and $\mathcal{A}^* \supset T^*$.

This ($*$)-extension is called correct, if an operator $\mathcal{A}_R = \frac{1}{2}(\mathcal{A} + \mathcal{A}^*)$ is a strong self-adjoint bi-extension of an operator $\mathcal{A}$. It is easy to show that if $\mathcal{A}$ is a ($*$)-extension of $T$, then $T$ and $T^*$ are quasi-kernels of $\mathcal{A}$ and $\mathcal{A}^*$, respectively.

Definition. We say that the operator $T$ of the class $\Omega_A$ belongs to the class $\Lambda_A$ if

1. $T$ admits a correct ($*$)-extension;
2. $A$ is the maximal common Hermitian part of $T$ and $T^*$.

Theorem 5. Let an operator $T$ belong to $\Omega_A$ and let $M$ be an operator in $[\mathcal{M}, \mathcal{M}]$ that is related to $T$ by Theorem 4. Then $T$ belongs to $\Lambda_A$ if and only if there exists either (1)-isometric operator or a ($\cdot$)-isometric operator $U$ in $[\mathcal{N}_i', \mathcal{N}_{-i}]$ such that

\[ \begin{align*}
(U + I)\mathcal{N}_i' + (M + I)(\mathcal{N}_i' \oplus \mathcal{N}) &= \mathcal{M}, \\
(U + I)\mathcal{N}_i' + (M + I)(\mathcal{N}_i' \oplus \mathcal{N}) &= \mathcal{M}.
\end{align*} \]
Corollary 2. If a closed Hermitian operator $A$ has finite and equal defect indices, then the class $\Omega_A$ coincides with the $\Lambda_A$.

Extended Resolvents and Extended Spectral Functions of a Hermitian Operator. Let $A$ be a closed Hermitian operator on $\mathcal{H}$ and $\mathfrak{h}$ be a Hilbert space such that $\mathcal{H}$ is a subspace of $\mathfrak{h}$. Let $\hat{A}$ be a self-adjoint extension of $A$ on $\mathfrak{h}$, and $\hat{E}(t)$ be the spectral function of $\hat{A}$. An operator function $R_\lambda = P_\mathcal{H}(\hat{A} - \lambda I)^{-1}|_\mathcal{H}$ is called a generalized resolvent of $A$, and $E(t) = P_\mathcal{H}\hat{E}(t)|_\mathcal{H}$ is the corresponding generalized spectral function. Here

$$R_\lambda = \int_{-\infty}^{\infty} \frac{dE(t)}{t - \lambda} \ (\text{Im}\lambda \neq 0).$$

If $\mathfrak{h} = \mathcal{H}$ then $R_\lambda$ and $E(t)$ are called canonical resolvent and canonical spectral function, respectively. According to [19] we denote by $\hat{R}_\lambda$ the $(-,\cdot)$-continuous operator from $\mathcal{H}_-$ into $\mathcal{H}$ which is adjoint to $R_\lambda$:

$$(\hat{R}_\lambda f, g) = (f, R_\lambda g) \quad (f \in \mathcal{H}_-, g \in \mathcal{H}).$$

It follows that $\hat{R}_\lambda f = R_\lambda f$ for $f \in \mathfrak{h}$, so that $\hat{R}_\lambda$ is an extension of $R_\lambda$ from $\mathcal{H}$ to $\mathcal{H}_-$ with respect to $(-,\cdot)$-continuity. The function $\hat{R}_\lambda$ of the parameter $\lambda$, $(\text{Im}\lambda \neq 0)$ is called the extended generalized (canonical) resolvent of the operator $A$. We write $\mathcal{K}$ for the family of all finite intervals on the real axis. It is known [19] that if $\Delta \in \mathcal{K}$ then $E(\Delta)\mathcal{H} \subset \mathcal{H}_+$ and the operator $E(\Delta)$ is $(\cdot,\cdot)$-continuous. We denote by $\hat{E}(\Delta)$ the $(-,\cdot)$-continuous operator from $\mathcal{H}_-$ to $\mathcal{H}$ that is adjoint to $E(\Delta) \in [\mathcal{H}_-, \mathcal{H}_+]$. Similarly,

$$(\hat{E}(\Delta)f, g) = (f, E(\Delta)g) \quad (f \in \mathcal{H}_-, g \in \mathcal{H}),$$

One can easily see that $\hat{E}(\Delta)f = E(\Delta)f$, $\forall f \in \mathcal{H}$, so that $\hat{E}(\Delta)$ is the extension of $E(\Delta)$ by continuity. We say that $\hat{E}(\Delta)$, as a function of $\Delta \in \mathcal{K}$, is the extended generalized (canonical) spectral function of $A$ corresponding to the self-adjoint extension $\hat{A}$ (or to the original spectral function $E(\Delta)$). It is known [19] that $\hat{E}(\Delta) \in [\mathcal{H}_-, \mathcal{H}_+]$, $\forall \Delta \in \mathcal{K}$, and $(\hat{E}(\Delta)f, f) \geq 0$ for all $f \in \mathcal{H}_-$. It is also known [19] that the complex scalar measure $(E(\Delta)f, g)$ is a complex function of bounded variation on the real axis. However, $(\hat{E}(\Delta)f, g)$ may be unbounded for $f, g \in \mathcal{H}_-$.

Now let $\hat{R}_\lambda$ be an extended generalized (canonical) resolvent of a closed Hermitian operator $A$ and let $\hat{E}(\Delta)$ be the corresponding extended generalized (canonical) spectral function. It was shown in [19] that for any $f, g \in \mathcal{H}_-$,

$$\int_{-\infty}^{+\infty} \frac{|d(\hat{E}(\Delta)f, g)|}{1 + t^2} < \infty,$$
and the following integral representation holds

\[
(24) \quad \hat{R}_\lambda - \frac{\hat{R}_i + \hat{R}_{-i}}{2} = \int_{-\infty}^{+\infty} \left( \frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d\hat{E}(t).
\]

**Lemma 6.** Let \( \mathbb{A} = A^* + \mathcal{R}^{-1}(S - \frac{i}{2} P^{+}_{\mathfrak{m}} + \frac{i}{2} P^{+}_{\mathfrak{n},-})P^{+}_{\mathfrak{m}} \) be a strong self-adjoint bi-extension of a regular Hermitian operator \( A \) with the quasi-kernel \( \hat{A} \) and let \( \hat{E}(\Delta) \) be the extended canonical spectral function of \( \hat{A} \). Then for every \( f \in \mathfrak{H} \oplus L, f \neq 0 \), and for every \( g \in \mathfrak{H} \) there is an integral representation

\[
(25) \quad (\hat{R}_\lambda f, g) = \int_{-\infty}^{+\infty} \left( \frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d(\hat{E}(t)f, g) + \frac{1}{2}((\hat{R}_i + \hat{R}_{-i})f, g).
\]

Here \( F = \mathfrak{H}_+ \ominus \mathfrak{D}(A), L = \mathcal{R}^{-1}(S - \frac{i}{2} P^{+}_{\mathfrak{m}} + \frac{i}{2} P^{+}_{\mathfrak{n},-})F, \hat{R}_\lambda = (\mathbb{A} - \lambda I)^{-1} \).

**Theorem 7.** Let \( \mathbb{A} = A^* + \mathcal{R}^{-1}(S - \frac{i}{2} P^{+}_{\mathfrak{m}} + \frac{i}{2} P^{+}_{\mathfrak{n},-})P^{+}_{\mathfrak{m}} \) be a strong self-adjoint bi-extension of a regular Hermitian operator \( A \) with the quasi-kernel \( \hat{A} \) and let \( \hat{E}(\Delta) \) be the extended canonical spectral function of \( \hat{A} \). Also, let \( F = \mathfrak{H} \ominus \mathfrak{D}(A) \) and \( L = \mathcal{R}^{-1}(S - \frac{i}{2} P^{+}_{\mathfrak{m}} + \frac{i}{2} P^{+}_{\mathfrak{n},-})F \). Then for every \( f \in L + \mathfrak{L} \) with \( f \neq 0 \) and \( f \in \mathcal{R}(\mathbb{A} - \lambda I) \), we have

\[
(26) \quad \int_{-\infty}^{+\infty} d(\hat{E}(t)f, f) = \infty, \quad \text{if} \quad f \notin \mathfrak{L},
\]

and

\[
(26') \quad \int_{-\infty}^{+\infty} d(\hat{E}(t)f, f) < \infty, \quad \text{if} \quad f \in \mathfrak{L}.
\]

Moreover, there exist real constants \( b \) and \( c \) such that

\[
(27) \quad c\|f\|_{\mathfrak{L}}^2 \leq \int_{-\infty}^{+\infty} \frac{d(\hat{E}(t)f, f)}{1 + t^2} \leq b\|f\|_{\mathfrak{L}}^2,
\]

for all \( f \in L + \mathfrak{L} \).

**Corollary 3.** In the settings of Theorem 7 for all \( f, g \in L + \mathfrak{L} \)

\[
(28) \quad \left| \left( \frac{\hat{R}_i + \hat{R}_{-i}}{2} f, g \right) \right| \leq a \sqrt{\int_{-\infty}^{+\infty} \frac{d(\hat{E}(t)f, f)}{1 + t^2} \cdot \sqrt{\int_{-\infty}^{+\infty} \frac{d(\hat{E}(t)g, g)}{1 + t^2}}},
\]

where \( a > 0 \) is a constant (see [2]).
3. Linear Stationary Conservative Dynamic Systems

In this section we consider linear stationary conservative dynamic systems (l. s. c. d. s.) \( \theta \) of the form

\[
\begin{cases}
(A - zI) = KJ\varphi_- \\
\varphi_+ = \varphi_- - 2iK^*x 
\end{cases}
\] (Im \( A = KJK^* \)).

In a system \( \theta \) of the form (29) \( A, K \) and \( J \) are bounded linear operators in Hilbert spaces, \( \varphi_- \) is an input vector, \( \varphi_+ \) is an output vector, and \( x \) is an inner state vector of the system \( \theta \). For our purposes we need the following more precise definition:

**Definition.** The array

\[
\theta = \begin{pmatrix}
\mathcal{H}_+ & A & K \\
\mathcal{H}_- & J & E
\end{pmatrix}
\] (30)

is called a linear stationary conservative dynamic system or Brodskić-Livšic rigged operator colligation if

1. \( A \) is a correct \((*)\)-extension of an operator \( T \) of the class \( \Lambda_A \).
2. \( J = J^* = J^{-1} \in [E, E], \quad \dim E < \infty \)
3. \( A - A^* = 2iKK^* \), where \( K \in [E, \mathcal{H}_-] \) \( (K^* \in [\mathcal{H}_+, E]) \)

In this case, the operator \( K \) is called a channel operator and \( J \) is called a direction operator. A system \( \theta \) of the form (30) will be called a scattering system (dissipative operator colligation) if \( J = I \). We will associate with the system \( \theta \) the operator-valued function

\[
W_{\theta}(z) = I - 2iK^*(A - zI)^{-1}KJ
\] (31)

which is called the transfer operator-valued function of the system \( \theta \) or the characteristic operator-valued function of Brodskić-Livšic rigged operator colligation. According to Theorem 7, \( \Re(K) \subset \Re(A - \lambda I) \) and therefore \( W_{\theta}(z) \) is well-defined. It may be shown [10], [25] that the transfer operator-function of the system \( \theta \) of the form (30) has the following properties:

\[
W_{\theta}^*(z)JW_{\theta}(z) - J \geq 0 \quad \text{(Im } z > 0, z \in \rho(T)),
\]

\[
W_{\theta}^*(z)JW_{\theta}(z) - J = 0 \quad \text{(Im } z = 0, z \in \rho(T)),
\]

\[
W_{\theta}^*(z)JW_{\theta}(z) - J \leq 0 \quad \text{(Im } z < 0, z \in \rho(T)),
\]

where \( \rho(T) \) is the set of regular points of an operator \( T \). Similar relations take place if we change \( W_{\theta}(z) \) to \( W_{\theta}^*(z) \) in (32). Thus, the transfer operator-valued function of the system
\( \theta \) of the form (30) is \( J \)-contractive in the lower half-plane on the set of regular points of an operator \( T \) and \( J \)-unitary on real regular points of an operator \( T \).

Let \( \theta \) be a l.s.c.d.s. of the form (30). We consider the operator-valued function

\[
V_{\theta}(z) = K^*(A_R - zI)^{-1}K.
\]

The transfer operator-function \( W_{\theta}(z) \) of the system \( \theta \) and an operator-function \( V_{\theta}(z) \) of the form (33) are connected with the relation

\[
V_{\theta}(z) = i[W_{\theta}(z) + I]^{-1}[W_{\theta}(z) - I]J.
\]

As it is known [11] an operator-function \( V(z) \in [E, E] \) is called an operator-valued \( R \)-function if it is holomorphic in the upper half-plane and \( \text{Im} \ V(z) \geq 0 \) whenever \( \text{Im} \ z > 0 \).

It is known [11,17] that an operator-valued \( R \)-function acting on a Hilbert space \( E \) \((\dim E < \infty)\) has an integral representation

\[
V(z) = Q + F \cdot z + \int_{-\infty}^{+\infty} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) dG(t),
\]

where \( Q = Q^* \), \( F \geq 0 \) in the Hilbert space \( E \), and \( G(t) \) is a non-decreasing operator-function on \((-\infty, +\infty)\) for which

\[
\int_{-\infty}^{+\infty} \frac{dG(t)}{1 + t^2} \in [E, E].
\]

**Definition.** We call an operator-valued \( R \)-function \( V(z) \) acting on a Hilbert space \( E \), \((\dim E < \infty)\) realizable if in some neighborhood of the point \((-i)\), the function \( V(z) \) can be represented in the form

\[
V(z) = i[W_{\theta}(z) + I]^{-1}[W_{\theta}(z) - I]J,
\]

where \( W_{\theta}(z) \) is the transfer operator-function of some l.s.c.d.s. \( \theta \) with the direction operator \( J \) \((J = J^* = J^{-1} \in [E, E])\).

**Definition.** An operator-valued \( R \)-function \( V(z) \in [E, E] \), \((\dim E < \infty)\) is said to be a member of the class \( N(R) \) if in the representation (35) we have

i) \( F = 0 \),

ii) \( Qe = \int_{-\infty}^{+\infty} \frac{t}{1 + t^2} dG(t)e \),

for all \( e \in E \) with

\[
\int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty.
\]

We now establish the next result.
**Theorem 8.** Let \( \theta \) be a l.s.c.d.s. of the form (30) with \( \dim E < \infty \). Then the operator-function \( V_\theta(z) \) of the form (33), (34) belongs to the class \( N(R) \).

**Proof.** Let \( G_{-i} \) be a neighborhood of \((-i)\) and \( \lambda, \mu \in G_{-i} \). Then,

\[
V_\theta(\lambda) - V_\theta(\mu) = K^*(A_R - \lambda I)^{-1}K - K^*(A_R - \mu I)^{-1}K
\]

and

\[
\frac{V_\theta(\lambda) - V_\theta(\mu)}{\mu - \lambda} = K^*(A_R - \lambda I)^{-1}(A_R - \mu I)^{-1}K,
\]

for all \( \lambda, \mu \in G_{-i} \). Therefore, letting \( \lambda \to \mu \) we can say that \( V_\theta(z) \) is holomorphic in \( G_{-i} \). Without loss of generality (see [25]) we can conclude that \( V_\theta(z) \) is holomorphic in any one of the half-planes.

It is obvious that \( V_\theta^*(z) = V_\theta(z) = V_\theta(\bar{z}) \). Furthermore,

\[
\text{Im}V_\theta(z) = \frac{1}{2i}K^*(A_R - \bar{z}I)^{-1}(A_R - zI)^{-1}K.
\]

Since \((-i)\) is a regular point of the operator \( T \) in the system (30) then (see [10]) \( I + iV(\lambda)J \) is invertible in \( G_{-i} \).

Let now \( D_z = (A_R - zI)^{-1}K \), then it is easy to see that the adjoint operator \( D_z^* \) is given by \( D_z^* = K^*(A_R - \bar{z}I)^{-1} \). Therefore, we have \( \text{Im}V_\theta(z) = \text{Im}zD_z^*D_z \) which implies that \( \text{Im}V_\theta(z) \geq 0 \) when \( \text{Im}z > 0 \). Hence we can conclude that \( V_\theta(z) \) is an operator \( R \)-function and admits representation (35).

Let now \( B = K^*(A_R + iI)^{-1}(A_R - iI)^{-1}K \). It follows from (39) that \( B = \frac{1}{2i}(V_\theta(i) - V_\theta^*(i)) \). Using Theorem 7 and representation (35) one can show that

\[
Bf = \int_{-\infty}^{\infty} \frac{dG(t)}{1 + t^2} f, \quad f \in E
\]

and \( B \in [E, E] \).

Let \( \hat{E}(\Delta) \) be the canonical extended spectral function of the quasi-kernel \( \hat{A} \) of the operator \( A_R = \frac{1}{2}(A + A^*) \). Then relying on Lemma 6 for all \( f, g \in E \) we have

\[
(V_\theta(\lambda)f, g)_E = \int_{-\infty}^{+\infty} \left( \frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d(G(t) f, g)_E + (\hat{Q}f, g)_E,
\]
where $\hat{G}(\Delta) = K^* \hat{E}(\Delta) K$, $\Delta \in \mathbb{R}$ and

$$\hat{Q} = \frac{1}{2} K^* [(\mathcal{A}_R - iI)^{-1} + (\mathcal{A}_R + iI)^{-1}] K = \frac{1}{2} [V_\theta(-i) + V_\theta^*(-i)].$$

From Theorem 7 (see also [19]), we have for all $f \in E$ with $Kf \in \mathcal{L}$,

$$\int_{-\infty}^{\infty} d(\hat{G}(t)f, f)_E < \infty,$$

and

$$c\|Kf\|^2 \leq \int_{-\infty}^{\infty} \frac{d(\hat{G}(t)f, f)_E}{1 + t^2} \leq b\|Kf\|^2.$$

Moreover, (28) implies that

$$\left| (\hat{Q}f, g)_E \right| \leq C \sqrt{\int_{-\infty}^{\infty} \frac{d(\hat{G}(t)f, f)_E}{1 + t^2}} \cdot \sqrt{\int_{-\infty}^{\infty} \frac{d(\hat{G}(t)g, g)_E}{1 + t^2}}.$$

By (41) we have for any $f, g \in E$

$$(V_\theta(\lambda)f, g)_E = (\hat{Q}f, g)_E + \int_{-\infty}^{\infty} \left( \frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d(\hat{G}(t)f, g)_E.$$

On the other hand (35) implies

$$(V_\theta(\lambda)f, g)_E = (Qf, g)_E + \lambda(Ff, g)_E + \int_{-\infty}^{\infty} \left( \frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d(G(t)f, g)_E.$$

Comparing (46) and (47) we get $(Qf, g)_E = (\hat{Q}f, g)_E$, $(Ff, g)_E = 0$, and $(G(\Delta)f, g) = (\hat{G}(\Delta)f, g)$ $(\Delta \in \mathbb{R})$, for all $f, g \in E$. Taking into account the continuity and positivity of $F$, $G(\Delta)$, and $\hat{G}(\Delta)$, we find that $F = 0$ and $G(\Delta) = \hat{G}(\Delta)$ $(\Delta \in \mathbb{R})$.

Thus,

$$V(\lambda) = Q + \int_{-\infty}^{\infty} \left( \frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) dG(t),$$

holds.
Let $E_{\infty} = K^{-1} \mathcal{L}, E_{\infty} \subset E$. Since $\hat{E}(\Delta)$ coincides with $E(\Delta)$ on $\mathcal{L}$, then for any $e \in E_{\infty}$, we have

$$\int_{-\infty}^{+\infty} d(\hat{G}(t)e, e)_E < \infty.$$

If $e \notin E_{\infty}$, then $Ke \notin \mathcal{L}$ (see Theorem 7) and

$$\int_{-\infty}^{+\infty} d(\hat{G}(t)e, e)_E = \infty.$$

Further, since

$$Q = \frac{1}{2} [V_0(i) + V_0(-i)] = \frac{1}{2} \left[K^*((A_R + iI)^{-1} + (A_R - iI)^{-1})K\right],$$

we have $\mathcal{A}(Q) \subseteq \mathcal{A}(K^*) \subseteq E$. Now formula (45) yields

$$|(Qf, g)_E| \leq C\|f\|_E \cdot \|g\|_E, \quad f, g \in E.$$

On the other hand, if $e \in E_{\infty}$ then

$$Qe = \frac{1}{2} \left[K^*(\hat{A}_R + iI)^{-1} + (\hat{A}_R - iI)^{-1})Ke\right]$$

$$= K^* \int_{-\infty}^{+\infty} \frac{t}{1 + t^2} dE(t)Ke = \int_{-\infty}^{+\infty} \frac{t}{t^2 + 1} d\hat{G}(t)e.$$

This completes the proof.

Next, we establish the converse.

**Theorem 9.** Let an operator-valued function $V(z)$ act on a finite-dimensional Hilbert space $E$ and belong to the class $N(R)$. Then $V(z)$ admits a realization by the system $\theta$ of the form (30) with a preassigned direction operator $J$ for which $I + iV(-i)J$ is invertible.

**Proof.** We will use several steps to prove this theorem.

**Step 1.** Let $C_{00}(E, (-\infty, +\infty))$ be the set of continuous compactly supported vector-valued functions $f(t)$ ($-\infty < t < +\infty$) with values in a finite dimensional Hilbert space $E$. We introduce an inner product $(\cdot, \cdot)$ defined by

$$\int_{-\infty}^{+\infty} (G(dt)f(t), g(t))_E$$

---

2The method of rigged Hilbert spaces for solving inverse problems in the theory of characteristic operator-valued functions was introduced in [23] and was developed further in [2].
for all \( f, g \in C_{00}(E, (-\infty, +\infty)) \). In order to construct a Hilbert space, we identify with zero all functions \( f(t) \) such that \((f, f) = 0\). Then we make the completion and obtain the new Hilbert space \( L^2_G(E) \). Let us note that the set \( C_{00}(E, (-\infty, +\infty)) \) is dense in \( L^2_G(E) \).

Moreover, if \( f(t) \) is continuous and
\[
\int_{-\infty}^{+\infty} (G(dt)f(t), f(t))_E < \infty,
\]
then \( f(t) \) belongs to \( L^2_G(E) \).

Let \( D_0 \) be the set of the continuous vector-valued (with values in \( E \)) functions \( f(t) \) such that in addition to (54), we have
\[
\int_{-\infty}^{+\infty} t^2(G(dt)f(t), f(t))_E < \infty.
\]

Since \( C_{00} \subset D_0 \), it follows that \( D_0 \) is dense in \( L^2_G(E) \). We introduce an operator \( \hat{A} \) on \( D_0 \) in the following way:
\[
(56) \quad \hat{A}f(t) = tf(t).
\]

Below we denote again by \( \hat{A} \) the closure of the Hermitian operator \( \hat{A} \) (56). It is easy to see that this operator is Hermitian. Now \( \hat{A} \) is a self-adjoint operator in \( L^2_G(E) \) (see [9]).

Let \( \tilde{H}_+ = D(\hat{A}) \) and define the inner product
\[
(57) \quad (f, g)_{\tilde{H}_+} = (f, g) + (\hat{A}f, \hat{A}g)
\]
for all \( f, g \in \tilde{H}_+ \). It is clear that \( \tilde{H}_+ \) is a Hilbert space with norm \( \| \cdot \|_{\tilde{H}_+} \) generated by the inner product (57). We equip the space \( L^2_G(E) \) with spaces \( \tilde{H}_+ \) and \( \tilde{H}_- \):
\[
(58) \quad \tilde{H}_+ \subset L^2_G(E) \subset \tilde{H}_-.
\]

Let us denote by \( \tilde{R} \) the corresponding Riesz-Berezanskii operator, \( \tilde{R} \in \tilde{[H}_-, \tilde{H}_+] \).

Consider the following subspaces of the space \( E \):
\[
(59) \quad E_\infty = \{ e \in E : \int_{-\infty}^{+\infty} d(G(t)e, e)_E < \infty \},
\]
\[
F_\infty = E_\infty^\perp.
\]

If \( e \in E_\infty \), then (54) implies that the function \( e(t) = e \) is an element of the space \( L^2_G(E) \). On the other hand, if \( e \in E \) and \( e \notin E_\infty \) then \( e(t) \) does not belong to \( L^2_G(E) \). It can be
shown that any function $e(t) = e \in E$ can be identified with an element of $\tilde{H}_-$. Indeed, since for all $e \in E$

\begin{equation}
\int_{-\infty}^{+\infty} \frac{d(G(t)e, e)_E}{1 + t^2} < \infty,
\end{equation}

the function

\begin{equation}
\tilde{e}(t) = \frac{e}{\sqrt{1 + t^2}}
\end{equation}

belongs to the space $L^2_G(E)$. Letting $f(t) \in \mathcal{D}_0$, we have

\begin{equation}
\int_{-\infty}^{+\infty} (1 + t^2)(G(dt)f(t), f(t))_E < \infty.
\end{equation}

Therefore, the function $\tilde{f}(t) = \sqrt{1 + t^2}f(t)$ belongs to the space $L^2_G(E)$ and hence

\begin{equation}
(f(t), \tilde{e}(t)) = \int_{-\infty}^{+\infty} (G(dt)\tilde{f}(t), \tilde{e}(t))_E.
\end{equation}

Furthermore,

\begin{equation}
|(f(t), \tilde{e}(t))| \leq \|f(t)\| \cdot \|\tilde{e}(t)\|
\end{equation}

\begin{equation}
= \sqrt{\int_{-\infty}^{+\infty} (1 + t^2)(G(dt)f(t), f(t))_E} \cdot \sqrt{\int_{-\infty}^{+\infty} \frac{d(G(t)\tilde{e}(t), \tilde{e}(t))_E}{1 + t^2}} e
\end{equation}

\begin{equation}
= \|f\|_{\tilde{H}_+} \cdot \|e\|_E.
\end{equation}

Also,

\begin{equation}
\int_{-\infty}^{+\infty} (G(dt)f(t), e(t))_E = \int_{-\infty}^{+\infty} \left(\sqrt{1 + t^2}G(dt)f(t), \frac{e}{\sqrt{1 + t^2}}\right)_E
\end{equation}

\begin{equation}
= \int_{-\infty}^{+\infty} (G(dt)\tilde{f}(t), \tilde{e}(t))_E
\end{equation}

\begin{equation}
= (f(t), \tilde{e}(t)).
\end{equation}

Therefore,

\begin{equation}
e(f) = \int_{-\infty}^{+\infty} (G(dt)f(t), e(t))_E
\end{equation}

is a continuous linear functional over $\tilde{H}_+$, for $f \in \mathcal{D}_0$. Since $\mathcal{D}_0$ is dense in $\tilde{H}_+$, $e(t) = e$

belongs to $\tilde{H}_-$. 

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We calculate the Riesz-Berezanskii mapping on the vectors \( e(t) = e, e \in E \). By the definition of \( \tilde{\mathcal{R}} \), for all \( f \in \tilde{\mathcal{H}}_+ \) we have \((f, e) = (f, \tilde{\mathcal{R}}e)_{\tilde{\mathcal{H}}_+}\). Hence, for all \( f \in \mathcal{D}_0 \) (see also [2])

\[
(f, e) = \int_{-\infty}^{+\infty} (G(dt)f(t), e(t))_E = \int_{-\infty}^{+\infty} (1 + t^2) \left( G(dt)f(t), \frac{e(t)}{1 + t^2} \right)_E
\]

Thus

\[
(65) \quad \tilde{\mathcal{R}}e = \frac{e(t)}{1 + t^2}, \quad e \in E.
\]

Let us note some properties of the operator \( \hat{\mathcal{A}} \). It is easy to see that for all \( g \in \tilde{\mathcal{H}}_+ \), we have that \( \|\hat{\mathcal{A}}g\| \leq \|g\|_{\tilde{\mathcal{H}}_+} \). Taking this into account we obtain

\[
(66) \quad \|\hat{\mathcal{A}}f\|_{\tilde{\mathcal{H}}_-} = \sup_{g \in \tilde{\mathcal{H}}_+} \frac{|(\hat{\mathcal{A}}f, g)|}{\|g\|_{\tilde{\mathcal{H}}_+}} = \sup_{g \in \tilde{\mathcal{H}}_+} \frac{|(f, \hat{\mathcal{A}}g)|}{\|g\|_{\tilde{\mathcal{H}}_+}} \leq \sup_{g \in \tilde{\mathcal{H}}_+} \frac{\|f\| \cdot \|\hat{\mathcal{A}}g\|}{\|g\|_{\tilde{\mathcal{H}}_+}} \leq \|f\|.
\]

Hence, the operator \( \hat{\mathcal{A}} \) is \((\cdot, \cdot)\)-continuous. Let \( \overline{\mathcal{A}} \) be the extension of the operator \( \hat{\mathcal{A}} \) to \( \mathcal{H} \) with respect to \((\cdot, \cdot)\)-continuity. Now,

\[
(67) \quad (\overline{\mathcal{A}} - \lambda I)^{-1}g - (\overline{\mathcal{A}} - \mu I)^{-1}g = (\lambda - \mu)(\overline{\mathcal{A}} - \lambda I)^{-1}(\overline{\mathcal{A}} - \mu I)^{-1}g
\]

holds for all \( g \in \tilde{\mathcal{H}}_- \). Note in particular that

\[
(68) \quad (\overline{\mathcal{A}} - iI)^{-1}g - (\overline{\mathcal{A}} + iI)^{-1}g = 2i(\overline{\mathcal{A}} - iI)^{-1}(\overline{\mathcal{A}} + iI)^{-1}g
\]

and

\[
(69) \quad \|((\overline{\mathcal{A}} - iI)^{-1}g\|^2 = \|((\overline{\mathcal{A}} + iI)^{-1}g\|^2
\]

for all \( g \) in \( \tilde{\mathcal{H}}_- \). It follows from (60) that the element

\[
(70) \quad f(t) = \frac{f}{t - \lambda}, \quad f \in E
\]

belongs to the space \( L^2_G(E) \). It is easy to show that, for all \( e \in E \),

\[
(71) \quad (\overline{\mathcal{A}} - \lambda I)^{-1}e = \frac{e}{t - \lambda}, \quad (\text{Im}\lambda \neq 0).
\]
Step 2. Now let $\tilde{H}_+$ be the Hilbert space constructed in Step 1 and let

$$D(A) = \tilde{H}_+ \ominus \tilde{R}E,$$

where by $\ominus$ we mean orthogonality in $\tilde{H}_+$. We define an operator $A$ on $D(A)$ by the following expression:

$$A = \hat{A} \bigg|_{D(A)}.$$

Obviously, $A$ is a closed Hermitian operator.

Let us note that if $E_\infty = 0$ then $D(A)$ is dense in $L^2_G(E)$. Define $H_0 = D(A)$ and let $P$ be the orthogonal projection of $\tilde{H} = L^2_G(E)$ onto $\tilde{H}$. We shall show that $PA$ and $P\hat{A}$ are closed operators in $H$.

The following obvious inclusions hold: $A \subset A_1 \subset \hat{A}$. It is easy to see that $D(A_1) = D(A) \oplus \tilde{RF}_\infty$, $D(A_1) = H_0$ and $A_1$ is a closed Hermitian operator. Indeed, if we identify the space $E$ with the space of functions $e(t) = e$, $e \in E$ we would obtain $L^2_G(E) \ominus H_0 = E_\infty$. Since

$$\int_{-\infty}^{+\infty} \frac{d(G(t)e, h)_E}{1 + t^2} = 0$$

and

$$\tilde{R} \tilde{e} = \frac{\tilde{e}}{1 + t^2}, \quad \tilde{e} \in F_\infty$$

for all $e \in E_\infty$, $h \in F_\infty$, we find that $E_\infty$ is $(\cdot)$-orthogonal to $RF_\infty$ and hence $\overline{D(A_1)} = H_0$.

We denote by $A_1^*$ the adjoint of the operator $A_1$. Now we are going to find the defect subspaces $\mathfrak{N}_i$ and $\mathfrak{N}_{-i}$ of the operator $A$. Since the subspace $E \in \tilde{H}_-$ is $(\cdot)$-orthogonal to $D(A)$, we have that $(\hat{A} \pm iI)^{-1}E = \mathfrak{N}_\pm i$. Moreover, by (71) we have

$$\overline{(\hat{A} \pm iI)^{-1}E} = \mathfrak{N}_\pm i.$$
Obviously, $\mathfrak{H}_0^0 \subset \mathfrak{D}_0$ because

$$\int_{-\infty}^{+\infty} \frac{t}{|t-\lambda|^2} (G(dt)e,e)_E \leq K(\lambda) \int_{-\infty}^{+\infty} (G(dt)e,e)_E < \infty, \ e \in E_\infty.$$ 

Taking into account that

(78) \[ \mathfrak{D}(A_1^*) = \mathfrak{D}(A) + \mathfrak{H}_i^0 + \mathfrak{H}_{-i}^0, \]

we can conclude that $\mathfrak{D}(A_1^*) \subseteq \mathfrak{D}(\hat{A})$. At the same time, the inclusion $A_1 \subset \hat{A}$ implies that $\mathfrak{D}(A_1^*) \supset \mathfrak{D}(\hat{A})$. Combining these two we obtain $\mathfrak{D}(A_1^*) = \mathfrak{D}(\hat{A})$ and $P\hat{A} = A_1^*$. Since $A_1^*$ is a closed operator, $P\hat{A}$ is also closed. Consequently, $\hat{A}$ is the regular self-adjoint extension of the operator $A$ which implies $A$ is a regular Hermitian operator.

Since $\hat{A}$ is the self-adjoint extension of operator $A$ we find by (10) that

(79) \[ \mathfrak{D}(\hat{A}) = \mathfrak{D}(A) + (I-U)\mathfrak{H}_i \]

for some admissible isometric operator $U$ acting from $\mathfrak{H}_i$ into $\mathfrak{H}_{-i}$. It is easy to check that $U(\hat{A} - iI)^{-1}e = (\hat{A} + iI)^{-1}e$, for all $e$ in $E$. Consequently, the operator $U$ has the form:

(80) \[ U\left(\frac{e}{t-i}\right) = \frac{e}{t+i}, \ e \in E. \]

Straightforward calculations show that

$$\hat{A}(I-U)\left(\frac{e}{t-i}\right) = \frac{te}{t-i} - \frac{e}{t+i} = \frac{2ite}{t^2+1}.$$ 

Let $A^*$ be the adjoint of the operator $A$. In the space $\mathfrak{D}(A^*) = \mathfrak{H}_+$ we introduce an inner product

(81) \[ (f,g)_+ = (f,g) + (A^*f,A^*g), \]

and construct the rigged space $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$ with corresponding Riesz-Berezanskii operator $\mathcal{R}$. Since $P\hat{A}$ is a closed Hermitian operator, $\mathfrak{H}_+$ is a subspace of $\mathfrak{H}_+$.

By Theorem 2, $\mathfrak{H}_+ = \mathfrak{D}(\hat{A}) + (U-I)\mathfrak{H}_i$, where

$$\mathfrak{H}_i = \{ f_i \in \mathfrak{H}_i, \ (U-I)f_i \in \mathfrak{H}_0 \}.$$ 

Taking into account that

$$\left( U-I \right) \left( \frac{e}{t-i} \right) = -\frac{2ie}{t^2+1}, \ e \in E,$$
we can conclude that
\[
\hat{\mathcal{N}}_i = \left\{ \frac{\tilde{e}}{t - \tilde{t}}, \ e \in F_\infty = E \ominus E_\infty \right\}.
\]

Therefore,
\[
(82) \quad \mathcal{D}(A^*) = \mathcal{D}(\hat{A}) + \left\{ \frac{t\tilde{e}}{t^2 + 1} \right\}, \ e \in F_\infty.
\]

**Step 3.** In this Step we will construct a special self-adjoint bi-extension whose quasi-kernel coincides with the operator \( \hat{A} \). Then applying (7), we will have
\[
\mathcal{H}_+ = \mathcal{D}(A) \oplus \mathcal{N}_i' \oplus \mathcal{N}'_{-i} \oplus \mathcal{N},
\]
where \( \mathcal{N}_{\pm i} \) are semidefect spaces of the operator \( A \), \( \mathcal{N} = \mathcal{R}E_\infty \), and
\[
\mathcal{D}(A) \oplus E_\infty = \mathcal{H} = L^2_G(E).
\]

We begin by setting
\[
(83) \quad (f, g)_1 = (f, g)_+ + (P_{\mathcal{H}_+} f_+, P_{\mathcal{H}_+} g)_+, \ \text{for all} \ f, g \in \mathcal{H}_+.
\]

Here \( P_{\mathcal{H}_+} \) is an orthoprojection of \( \mathcal{H}_+ \) onto \( \mathcal{H} \). Obviously, the norm \( \| \cdot \|_1 \) is equivalent to \( \| \cdot \|_+ \). We denote by \( \mathcal{H}_{+1} \) the space \( \mathcal{H}_+ \) with the norm \( \| \cdot \|_1 \), so that \( \mathcal{H}_{+1} \subset \mathcal{H} \subset \mathcal{H}_{-1} \) is the corresponding rigged space with Riesz-Berezanskii operator \( \mathcal{R}_1 \).

By Theorem 1 there exists a (1)-isometric operator \( V \) such that
\[
(84) \quad \mathcal{D}(\hat{A}) = \mathcal{D}(A) \oplus (V + I)(\mathcal{N}_i' \oplus \mathcal{N}),
\]
where \( \mathcal{D}(V) = \mathcal{N}_i' \oplus \mathcal{N} \), \( \mathcal{R}(V) = \mathcal{N}_{-i} \oplus \mathcal{N} \) and \( (-1) \) is a regular point for the operator \( V \). Moreover,
\[
(85) \quad \begin{cases} 
\varphi = i(I + P_{\mathcal{H}_{+1}'})(A^* + iI)^{-1}f_i, \\
V\varphi = i(I + P_{\mathcal{H}_{-1}'})(A^* - iI)^{-1}Uf_i,
\end{cases}
\]
where \( \varphi \in \mathcal{D}(V) \), \( f_i \in \mathcal{N}_i \).

Here \( U \) is the isometric operator described in Step 2. Consequently we obtain
\[
(86) \quad \begin{cases} 
f_i = \frac{i}{2}(A^* + iI)(I + P_{\mathcal{H}_{+1}'})\varphi, \\
Uf_i = -\frac{i}{2}(A^* - iI)(I + P_{\mathcal{H}_{+1}'})V\varphi,
\end{cases}
\]
where \( \varphi \in \mathcal{D}(V) \), \( f_i \in \mathcal{N}_i \).
It follows that

\[ f_i - Uf_i = \varphi + V\varphi + iA^*P_{\Omega}^+(V - I)\varphi \]

\[ \hat{A}(f_i - Uf_i) = i(I + U)f_i = A^*(\varphi + V\varphi) + iP_{\Omega}^+(I - V)\varphi \]

\[ f_i + Uf_i = \varphi - V\varphi - iA^*P_{\Omega}^+(I - V)\varphi \]

Applying formula (11) we get

\[ \mathfrak{H}_+ = \mathfrak{D}(\hat{A}) + (U + I)\mathfrak{N}_i, \text{ and } \mathfrak{N}_i = \{ f_i \in \mathfrak{N}_i \mid (U - I)f_i \in \mathfrak{H} \}. \]

Since \( f_i - Uf_i = \varphi + V\varphi + iA^*P_{\Omega}^+(V - I)\varphi \), we find that \( f_i - Uf_i \in \mathfrak{H} \) if and only if \( P_{\Omega}^+(V + I)\varphi = 0 \). (This follows from the fact that \( A^*P_{\Omega}^+(V - I)\varphi \in \mathfrak{D}(A) \subset \mathfrak{H} \) and from the formula \( \mathfrak{H} = \mathfrak{H}_0 + \mathfrak{N} \) (see [4])). Let us note that if \( P_{\Omega}^+(V + I)\varphi = 0 \) then \( f_i + Uf_i = \varphi - V\varphi \). Thus,

(87) \[ \mathfrak{N}_i = \{ f = (A^* + iI)(I + P_{\Omega}^+)\varphi, \quad P_{\Omega}^+(V + I)\varphi = 0 \}. \]

Let \( N = \ker P_{\Omega}^+(I + V) \). Then we have

(88) \[ \mathfrak{H}_+ = \mathfrak{D}(\hat{A}) + (I - V)N. \]

We denote by \( P_0 \) the projection operator of \( \mathfrak{H}_+ \) onto \( \mathfrak{D}(\hat{A}) \) along \( (I - V)N \), \( P_1 = I - P_0 \). Since \( \mathfrak{D}(\hat{A}) = \mathfrak{H}_+ \), we have \( P_0 \in [\mathfrak{H}_+, \mathfrak{H}_+] \). We will denote by \( P_0^* \in [\mathfrak{H}_-, \mathfrak{H}_-] \) the adjoint operator to \( P_0 \), i.e. \( (P_0f, g) = (f, P_0^*g) \), for all \( f \in \mathfrak{H}_+, g \in \mathfrak{H}_- \). If \( \tilde{f}_i \in \mathfrak{N}_i \), then

\[ \tilde{f}_i + U\tilde{f}_i = (I - V)\varphi, \text{ for } \varphi \in N, \text{ and} \]

\[ A^*(I - V)\varphi = iP_{\Omega}^+\varphi + iP_{\Omega}^+V\varphi + AP_{\Omega}^+(I - V)\varphi = i(V + I)\varphi + A^*P_{\Omega}^+(I - V)\varphi \]

\[ = i[(I + V)\varphi - iA^*P_{\Omega}^+(I - V)\varphi]. \]

This implies

\[ A^*(I + U)\tilde{f}_i = i(\tilde{f}_i - U\tilde{f}_i). \]

Hence

(89) \[ A^* \left( \frac{\tilde{e}}{t^2 + 1} \right) = - \frac{\tilde{e}}{t^2 + 1}, \quad \tilde{e} \in F_{\infty}. \]

Let \( Q \in [E, E] \) be the operator in the definition of the class \( N(R) \). We introduce a new operator \( R_0 \) acting in the following way:

(90) \[ R_0f = iQ\tilde{R}^{-1}A^*P_1f, \quad f \in \mathfrak{H}_+. \]
In order to show that $R_0 \in [\mathcal{H}_+, E]$, we consider the following calculation for $f \in \mathcal{H}_+$:

$$\|R_0 f\|_E = \sup_{g \in E} \frac{|(R_0 f, g)_E|}{\|g\|_E} = \sup_{g \in E} \frac{|(Q \tilde{R}^{-1} A^* P_1 f, g)_E|}{\|g\|_E} = \sup_{g \in E} \frac{|(\tilde{R}^{-1} A^* P_1 f, Qg)_E|}{\|g\|_E} \leq \sup_{g \in E} \frac{\|\tilde{R}^{-1} A^* P_1 f\|_E \cdot \|Qg\|_E}{\|g\|_E} \leq c \|A^* P_1 f\|_{\tilde{H}_+} \leq b \|A^* P_1 f\|_{\tilde{H}_+}, \quad b, c - \text{constants.}$$

Here we used that $P_1 f \subset \mathcal{D}(\hat{A})$, for all $f \in \tilde{H}_+$, formulas (65) and (89), and the equivalence of the norms $\| \cdot \|_{\tilde{H}_+}$ and $\| \cdot \|_+.

For $f \in \tilde{H}_+$, we have $P_1 f = (I - V)\varphi$, $\varphi \in N$ and

$$A^* P_1 f = i(V + I)\varphi + iA^* P_{\tilde{R}}^+(V - I)\varphi.$$ 

We now have

$$\|A^* P_{\tilde{R}}^+(V - I)\varphi\|_+^2 = \|A^* P_{\tilde{R}}^+(V - I)\varphi\|^2 + \|A^* A^* P_{\tilde{R}}^+(V - I)\varphi\|^2 = \|A^* P_{\tilde{R}}^+(V - I)\varphi\|^2 + \|P_{\tilde{R}}^+(V - I)\varphi\|^2 = \|P_{\tilde{R}}^+(V - I)\varphi\|^2,$$

and

$$\|i(V + I)\varphi + iA^* P_{\tilde{R}}^+(V - I)\varphi\|_+^2 = \|A^* P_{\tilde{R}}^+(V - I)\varphi\|^2 + \|\varphi + V\varphi\|^2 \leq \|P_{\tilde{R}}^+(V - I)\varphi\|^2 + \|\varphi + V\varphi\|^2 = \|\varphi - V\varphi\|^2.$$ 

This implies that there exists a constant $k$ such that

$$\|A^* P_1 f\| \leq \|P_1 f\|_+ \leq k\|f\|_+, \quad \forall f \in \mathcal{H}_+. \quad (91)$$

Therefore, for some constant $d > 0$ we have $\|R_0 f\| \leq d\|f\|_+, \quad \forall f \in \mathcal{H}_+$. Thus, $R_0 \in [\mathcal{H}_+, E]$.

Let $R_0^*$ be the adjoint operator to $R_0$, i.e. $R_0^* \in [E, \mathcal{H}_+]$ and for all $f \in \mathcal{H}_+$, $e \in E$, $(R_0 f, e)_E = (f, R_0^* e)$. Since $R_0(\mathcal{D}(\hat{A})) = 0$, $\mathcal{H}(R_0^*)$ is ($\cdot$)-orthogonal to $\mathcal{D}(\hat{A})$. Letting $\mathcal{M} = \mathcal{M}_{-i} + \mathcal{N}' \oplus \mathcal{M}$, we obtain from (88)

$$\mathcal{M} = (V + I)(\mathcal{M}' \oplus \mathcal{M}) + (I - V)N. \quad (92)$$

In the space $\mathcal{M}$ we define an operator $S$ in the following way

$$S(\varphi + V\varphi) = \frac{i}{2}(I - V)\varphi, \quad \varphi \in \mathcal{M}' \oplus \mathcal{M}, \quad (93)$$

$$S(\varphi_N - V\varphi_N) = \left[-R_1(R_0^* + P_0^*)\tilde{R}^{-1} A^* + \frac{i}{2}(P_{\tilde{R}^*}^+ - P_{\tilde{R}_-}^+ \right)(\varphi_N - V\varphi_N),$$

$$76$$
where \( \varphi_N \in N \). In order to show that \( S \) is a (1)-self-adjoint operator on \( \mathcal{M} \), we first check that

\[
(S(\varphi + V \varphi), \varphi + V \varphi) = (\varphi + V \varphi, S(\varphi + V \varphi)), \quad \varphi \in \mathcal{N}_i \oplus \mathcal{N}.
\]

(94)

It is easy to see that

\[
(P_{\mathcal{N}_i}^+ - P_{\mathcal{N}_i'}^+)(\varphi_N - V \varphi_N) = \varphi_N + V \varphi_N, \quad \varphi_N \in N.
\]

This follows from the definition of the space \( N \) and the fact that \( \varphi_N \) belongs to \( \mathcal{N}_i'. \). Furthermore, since \( \varphi_N \in \mathcal{N}_i' \), and \( V \varphi_N \in \mathcal{N}_i' \), we have that \( P_{\mathcal{N}_i}^+ \varphi_N = \varphi_N, P_{\mathcal{N}_i}^+ V \varphi_N = \varphi_N, \) and \( P_{\mathcal{N}_i}^+ V \varphi_N = P_{\mathcal{N}_i'}^+ \varphi_N = 0 \). Consequently,

\[
((\varphi_N + V \varphi_N), \varphi_N - V \varphi_N) = ||\varphi_N||^2 - ||V \varphi_N||^2
\]

(95)

\[
= ||P_{\mathcal{N}_i}^+ \varphi_N||^2 - ||P_{\mathcal{N}_i'}^+ V \varphi_N||^2 = 0.
\]

Since \( P_0(I - V)N = 0 \), we have

\[
(\mathcal{R}_1 P_0^* \mathcal{R}^{-1} A^*(\varphi_N - V \varphi_N), \varphi_N - V \varphi_N) = (\mathcal{R}^{-1} A^*(\varphi_N - V \varphi_N), P_0(\varphi_N - V \varphi_N)) = 0.
\]

(96)

This allows us to consider only the \( R_0^* \)-containing part of (93), i.e.

\[
(S(\varphi_N - V \varphi_N), \varphi_N - V \varphi_N) = (-\mathcal{R}_1 R_0^* \mathcal{R}^{-1} A^*(\varphi_N - V \varphi_N), (\varphi_N - V \varphi_N)_1 \nonumber
\]

\[
= (\mathcal{R}^{-1} A^*(\varphi_N - V \varphi_N), -R_0(\varphi_N - V \varphi_N)) E
\]

\[
= (\mathcal{R}^{-1} A^*(\varphi_N - V \varphi_N), iQ \mathcal{R}^{-1} A^* P_1(\varphi_N - V \varphi_N)) E
\]

\[
= (-iQ \mathcal{R}^{-1} A^*(\varphi_N - V \varphi_N), \mathcal{R}^{-1} A^*(\varphi_N - V \varphi_N)) E
\]

\[
= ((\varphi_N - V \varphi_N), R_0^* \mathcal{R}^{-1} A^*(\varphi_N - V \varphi_N)) E
\]

\[
= ((\varphi_N - V \varphi_N), \mathcal{R}_1 R_0^* \mathcal{R}^{-1} A^*(\varphi_N - V \varphi_N))_1
\]

\[
= ((\varphi_N - V \varphi_N), S(\varphi_N - V \varphi_N))_1.
\]

Now we will show that

\[
(S(\varphi + V \varphi), \varphi_N - V \varphi_N) = (\varphi + V \varphi, S(\varphi + V \varphi))_1, \quad \varphi_N \in N, \quad \varphi \in \mathcal{M}_i' \oplus \mathcal{M}.
\]

(97)

Let us note that \( P_{\mathcal{M}}^+(\varphi_N + V \varphi_N) = 0 \) implies \( P_{\mathcal{M}}^+ V \varphi_N = -P_{\mathcal{M}}^+ V \varphi_N \). Also, \( (\varphi_N)_1 = (V \varphi, V \varphi)_1 \), since \( V \) is a (1)-isometric mapping. We will now show that the orthogonality
relations yield \((\varphi, V\varphi_N)_1 = (\varphi, P^+_{\mathfrak{r}_1} V\varphi_N)_1 = 0\). First we need a calculation

\[
(S(\varphi + V\varphi), \varphi_N - V\varphi_N)_1 = i\frac{1}{2}((I - V)\varphi, \varphi_N - V\varphi_N)_1
\]

\[
= i(\varphi, \varphi_N)_1 - i\frac{1}{2}(\varphi, V\varphi_N)_1 - i\frac{1}{2}(V\varphi, \varphi_N)_1
\]

\[
= i(\varphi, \varphi_N)_1 - i\frac{1}{2}(\varphi, V\varphi_N)_1 - i\frac{1}{2}(\varphi, P^+_{\mathfrak{r}_1} V\varphi_N)_1
\]

\[
= i(\varphi, \varphi_N)_1 - i\frac{1}{2}(\varphi, V\varphi_N)_1 + i\frac{1}{2}(\varphi, P^+_{\mathfrak{r}_1} V\varphi_N)_1
\]

\[
= i(\varphi, \varphi_N)_1 + i\frac{1}{2}(P^+_{\mathfrak{r}_1}(I - V)\varphi, \varphi_N)_1.
\]

Also, note that

\[
\left(\varphi + V\varphi, i\frac{1}{2}(P^+_{\mathfrak{r}_1} - P^+_{\mathfrak{r}_1}) (\varphi_N - V\varphi_N)\right)_1 = -i\frac{1}{2} (\varphi + V\varphi, \varphi_N + V\varphi_N)_1,
\]

and

\[
(\varphi + V\varphi, S(\varphi_N - V\varphi_N)_1 = (\varphi + V\varphi, -\mathcal{R}_1(R^*_0 + P^*_0)\hat{\mathcal{R}}^{-1} A^*(\varphi_N - V\varphi_N))_1
\]

\[
- i\frac{1}{2}(\varphi + V\varphi, \varphi_N + V\varphi_N)_1.
\]

Next, recall that \(\mathcal{R}(R^*_0)\) is \((\cdot)\)-orthogonal to \(\mathcal{D}(\hat{A})\) and

\[
\varphi + V\varphi \in \mathcal{D}(\hat{A}) = \mathcal{D}(A) \oplus (V + I)(\mathcal{R}_i' \oplus \mathfrak{r}).
\]

It follows that

\[
(\varphi + V\varphi, \mathcal{R}_1 R^*_0 \hat{\mathcal{R}}^{-1} A^*(\varphi_N - V\varphi_N))_1 = (\varphi + V\varphi, R^*_0 \hat{\mathcal{R}}^{-1} A^*(\varphi_N - V\varphi_N))_1 = 0,
\]

\[
(\varphi + V\varphi, -\mathcal{R}_1 P^*_0 \hat{\mathcal{R}}^{-1} A^*(\varphi_N - V\varphi_N))_1 = -(\varphi + V\varphi, A^*(\varphi_N - V\varphi_N))_B
\]

\[
= -(\varphi + V\varphi, A^*(\varphi_N - V\varphi_N))
\]

\[
= -(\hat{A}(\varphi + V\varphi), \hat{A}_0 A^*(\varphi_N - V\varphi_N)).
\]

Applying Theorem 1 we obtain:

\[
\hat{A}(\varphi + V\varphi) = A^*(\varphi + V\varphi) + i\frac{1}{2} \mathcal{R}^{-1} P^+_{\mathfrak{r}_1} (I - V)\varphi,
\]

\[
A^*(\varphi_N - V\varphi_N) = i(I + V)\varphi_N + A^* P^+_{\mathfrak{r}_1} (I - V)\varphi_N.
\]

\[
\hat{A}A^*(\varphi_N - V\varphi_N) = AA^* P^+_{\mathfrak{r}_1} (I - V)\varphi_N + iA^*(V + I)\varphi_N - i\frac{1}{2} \mathcal{R}^{-1} P^+_{\mathfrak{r}_1} (I - V)\varphi_N
\]

\[
= iA^*(V + I)\varphi_N - P^+_{\mathfrak{r}_1} (I - V)\varphi_N.
\]
Here we used the following relations:

\[
A^*(I - V) \in \mathcal{D}(A),
\]

\[
\hat{A}(f_i - Uf_i) = A^*(\varphi + V\varphi) + iP_{\Theta}^{+}(I - V)\varphi,
\]

\[
f_i - Uf_i = \varphi + V\varphi + iA^*P_{\Theta}^{+}(V - I)\varphi,
\]

\[
\hat{A}(\varphi + V\varphi) = A^*(\varphi + V\varphi) + \frac{i}{2}R^{-1}P_{\Theta}^{+}(I - V)\varphi,
\]

and

\[
AA^*P_{\Theta}^{+}(I - V\varphi_N) - \frac{1}{2}R^{-1}(I - V)\varphi = -P_{\Theta}^{+}(I - V)\varphi_N.
\]

The above identities yield that

\[
(\varphi + V\varphi, A^*(\varphi_N - V\varphi_N))_{\delta^+_\Theta} = (\varphi + V\varphi, i(\varphi_N + V\varphi_N))_1 - i(P_{\Theta}^{+}(I - V)\varphi, \varphi_N)_1.
\]

Thus,

\[
(\varphi + V\varphi, -\mathcal{R}_1P_0^+\tilde{R}^{-1}A^*(\varphi_N - V\varphi_N))_0 = i(\varphi + V\varphi, \varphi_N + V\varphi_N)
\]

\[
+ i(P_{\Theta}^{+}(I - V)\varphi, \varphi_N),
\]

\[
(\varphi + V\varphi, \frac{i}{2}(\varphi_N + V\varphi_N))_1 = -\frac{i}{2}(\varphi + V\varphi, \varphi_N + V\varphi_N)_1,
\]

and

\[
(\varphi + V\varphi, S(\varphi_N - V\varphi_N))_1 = i(\varphi + V\varphi, \varphi_N + V\varphi_N)
\]

\[
+ i(P_{\Theta}^{+}(I - V)\varphi, \varphi_N)_1 - \frac{i}{2}(\varphi + V\varphi, \varphi_N + V\varphi_N)_1
\]

\[
= i(\varphi, \varphi_N)_1 + \frac{i}{2}(V\varphi, \varphi_N)_1
\]

\[
+ \frac{i}{2}(\varphi, V\varphi_N)_1 + i(P_{\Theta}^{+}(I - V)\varphi, \varphi_N)_1
\]

\[
= i(\varphi, \varphi_N)_1 + \frac{i}{2}(P_{\Theta}^{+}(I - V)\varphi, \varphi_N)_1
\]

\[
= (S(\varphi + V\varphi), \varphi_N - V\varphi_N).
\]

This shows that \( S \) is a (1)-self-adjoint operator in \( \mathfrak{M} \).

By Corollary 2, a self-adjoint bi-extension of the operator \( A \) is defined by the formula

\[
\mathcal{B} = AP_{\mathcal{D}(A)}^{+} + \left[ A^* + \mathcal{R}^{-1} \left( S - \frac{i}{2}P_{\mathfrak{U}^\prime} + \frac{i}{2}P_{\mathfrak{U}''} \right) \right] P_{\Theta}^{+},
\]

where \( S \) is defined by (97). Obviously, if \( f = f_A + (V + I)\varphi, \varphi \in \mathfrak{U}^\prime \oplus \mathfrak{M}, \) and \( f_A \in \mathcal{D}(A) \) then \( \mathcal{B}f = \hat{A}f \). This means that the quasi-kernel of the operator \( \mathcal{B} \) coincides with \( \hat{A} \).
Step 4. In this Step we will construct a \((\ast)\)-extension of some operator of the class \(\Lambda_A\). First, we introduce the bounded linear operator \(K\) acting from the space \(E\) into the space \(H\) as follows:

\[
Ke = (P_0^* + R_0^*)P_{F_\infty} + \hat{I}P_{E_\infty}e, \quad e \in E,
\]

where \(P_{F_\infty}\) and \(P_{E_\infty}\) are orthogonal projections of the space \(E\) onto \(F_\infty\) and \(E_\infty\) respectively, and \(\hat{I}\) is an embedding of \(E_\infty\) in \(H\).

Let \(K^* \in [\mathfrak{H}_+, E]\) be an adjoint of the operator \(K\), i.e.

\[
(Kf, g) = (f, K^*g), \quad f \in E, \; g \in \mathfrak{H}_+.
\]

Let

\[
\mathbb{C} = K^*JK,
\]

where \(J \in [E, E]\) satisfies \(J = J^* = J^{-1}\). Since \(\mathfrak{K}(K)\) is orthogonal to \(\mathfrak{D}(A), \mathbb{C}(\mathfrak{D}(A)) = 0\). Moreover, \((\mathbb{C}f, g) = (f, \mathbb{C}g)\) for all \(f \in \mathfrak{H}_+, g \in \mathfrak{H}_+\).

We define an operator \(\mathbb{A}\) by

\[
\mathbb{A} = \mathbb{B} + i\mathbb{C}.
\]

We now show that \(\mathbb{A}\) is a \((\ast)\)-extension of some operator \(T\) of the class \(\Lambda_A\).

Let \(\lambda\) be a regular point of the operator \(\hat{A}\) and let \(\hat{R}_\lambda = (\hat{B} - \lambda I)^{-1}\). Also, note that

\[
(\hat{R}_\lambda f, g) = (f, (\hat{A} - \overline{\lambda} I)^{-1}g), \quad \forall f \in \mathfrak{H}_-, \; g \in \mathfrak{H}.
\]

As it was shown in Step 1 (see (71))

\[
(\overline{A} - \lambda I)^{-1} = \frac{e}{t - \lambda}, \quad \forall e \in E,
\]

where \(E\) is considered as a subspace of \(\mathfrak{H}_-\). Clearly,

\[
(\hat{R}_\lambda P_0^* e, g) = (P_0^* e, (\hat{A} - \overline{\lambda} I)^{-1}g) = (e, (\hat{A} - \overline{\lambda} I)^{-1}g) = ((\overline{A} - \lambda I)^{-1}e, g), \quad \forall e \in E, \; g \in \mathfrak{H} = L_G^2(E).
\]

It follows that

\[
\hat{R}_\lambda P_0^* e = \frac{e}{t - \lambda}, \quad \forall e \in E.
\]
Since \( R_0(\mathcal{D}(\hat{A})) = 0, R_0(\hat{A} - \lambda I)^{-1}g = 0 \), for all \( g \in \mathcal{H} \), and we have
\[
(\hat{R}_\lambda R_0^*e, g) = (R_0^*e, (\hat{A} - \lambda I)^{-1}g) = (e, R_0(\hat{A} - \lambda I)^{-1}g) = 0,
\]
\[
\hat{R}_\lambda Ke_1 = \hat{R}_\lambda P_0^*e_1 = \frac{e_1}{t - \lambda}, \quad e_1 \in F_\infty,
\]
\[
\hat{R}_\lambda Ke_2 = \hat{R}_\lambda e_2 \frac{e_2}{t - \lambda} \in \mathcal{H}_+, \quad e_2 \in E_\infty,
\]
This implies that the operator \( K \) is invertible. Indeed, if \( Ke = 0 \), then \((P_0^* + R_0^*)e_1 = -\hat{I}e_2\) and \(\hat{R}_\lambda Ke = 0\). Hence, \(\hat{R}_\lambda (P_0^* + R_0^*)\hat{e} = -\hat{R}_\lambda e_2\). That is,
\[
\frac{e_1}{t - \lambda} = \frac{e_2}{t - \lambda}, \quad e = \hat{e} + e_1,
\]
which implies that \(e = 0\).

We should also note that \(\hat{R}_\lambda K \in [E, \mathcal{H}_+]\), since \(\hat{R}_\lambda\) maps \(\mathcal{R}(K)\) into \(\mathcal{H}_+\) continuously.

Let us consider now the operator-valued function \(V\) defined by

\[
(103) \quad \quad V(\lambda) = K^*\hat{R}_\lambda K, \quad \text{Im}\lambda \neq 0.
\]
Obviously, \((V(\lambda)e, h)_E = (\hat{R}_\lambda Ke, Kh)\) for \(e \in E, h \in E, e = e_1 + e_2, h = h_1 + h_2\). Therefore,
\[
(\hat{R}_\lambda Ke, Kh) = (\hat{R}_\lambda(P_0^* + R_0^*)e_1 + \hat{R}_\lambda e_2, (P_0^* + R_0^*)h_1 + \hat{I}h_2)
\]
\[
= (\hat{R}_\lambda P_0^*e_1 + \hat{R}_\lambda e_2, (P_0^* + R_0^*)h_1 + \hat{I}h_2)
\]
\[
= (\hat{R}_\lambda P_0^*e_1, P_0^*h_1) + (\hat{R}_\lambda P_0^*e_1, R_0^*h_1) + (\hat{R}_\lambda P_0^*e_1, h_2) + (\hat{R}_\lambda e_2, P_0^*h_1)
\]
\[
+ (\hat{R}_\lambda e_2, R_0^*h_2) + (\hat{R}_\lambda e_2, h_2)
\]
\[
= (P_0\hat{R}_\lambda P_0^*e_1, h_1) + (P_0\hat{R}_\lambda P_0^*e_1, h_2) + (\hat{R}_\lambda P_0^*e_1, h_2) + (\hat{R}_\lambda e_2, h_2)
\]
\[
+ (R_0\hat{R}_\lambda e_2, h_2)_E + (\hat{R}_\lambda e_2, h_2).
\]
We also have
\[
\hat{R}_\lambda P_0^*e_1 = \frac{e_1}{t - \lambda} \notin \mathcal{H}_-.
\]
Consider an element
\[
\frac{e_1}{t - \lambda} - \frac{te_1}{t^2 + 1} = -\frac{\lambda t e_1}{(t - \lambda)(t^2 + 1)}, \quad e_1 \in F_\infty.
\]
Clearly
\[
\int_{-\infty}^{+\infty} \frac{[\lambda^2 t^4]}{|t - \lambda|^2(t^2 + 1)} \frac{d(G(t)e_1, e_1)_E}{1 + t^2} < \infty,
\]
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and hence
\[ \frac{e_1}{t - \lambda} - \frac{te_1}{t^2 + 1} \in \mathcal{D}(\hat{A}). \]
Moreover,
\[ \frac{te_1}{t^2 + 1} \in (I - V)N, \quad e_1 \in F_\infty. \]
This implies
\[
\begin{align*}
P_0 \left\{ \frac{e_1}{t - \lambda} \right\} &= \frac{e_1}{t - \lambda} - \frac{te_1}{t^2 + 1}, \\
P_1 \left\{ \frac{e_1}{t - \lambda} \right\} &= \frac{te_1}{t^2 + 1}.
\end{align*}
\]
Consequently,
\[
(P_0 \hat{R}_\lambda P_0^* e_1, h_2) = \int_{-\infty}^{+\infty} \left( \frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) d(G(t)e_1, h_2)_E.
\]
We also have that
\[
(R_0 \hat{R}_\lambda P_0^*, h_1)_E = -(Q\hat{R}^{-1}A^*P_1\hat{R}_\lambda P_0^* e_1, h_1)_E = -(\hat{R}^{-1}A^*P_1\hat{R}_\lambda P_0 e_1, Qh_1)_E.
\]
From (65) and (89) we obtain
\[
\hat{R}^{-1}A^*P_1\hat{R}_\lambda P_0^* e_1 = \hat{R}^* \left( \frac{e_1}{t^2 + 1} \right) = -e_1,
\]
from which it follows that
\[
(R_0 \hat{R}_\lambda P_0^*, h_2)_E = (e_1, Qh_2)_E = (Qe_1, h_2)_E.
\]
Furthermore we obtain
\[
(\hat{R}_\lambda P_0^* e_1, h_2) = \int_{-\infty}^{+\infty} \left( \frac{1}{t - \lambda} \right) d(G(t)e_2, h_2)_E
\]
\[
= \int_{-\infty}^{+\infty} \left( \frac{1}{t - \lambda} \right) d(G(t)e_2, h_2)_E - (Qe_1, h_2)_E + (Qe_1, h_2)_E
\]
\[
= \int_{-\infty}^{+\infty} \frac{t}{t^2 + 1} d(G(t)e_1, h_2)_E + (Qe_1, h_2)_E
\]
\[
= \int_{-\infty}^{+\infty} \left( \frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) d(G(t)e_1, h_2)_E + (Qe_1, h_2)_E.
\]
Since $R_0 \hat{R}_\lambda e_2 = 0$, we have

$$
(\hat{R}_\lambda e_2, h_1) = \int_{-\infty}^{+\infty} \left( \frac{1}{t - \lambda} \right) d(G(t)e_2, h_1)_E - (Qe_2, h_1)_E + (Qe_2, h_1)_E
$$

$$
= \int_{-\infty}^{+\infty} \left( \frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) d(G(t)e_2, h_1)_E + (Qe_2, h_1)_E
$$

Thus,

$$
(\hat{R}_\lambda e_2, h_2) = \int_{-\infty}^{+\infty} \left( \frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) d(G(t)e_2, h_2)_E + (Qe_2, h_2)_E
$$

These calculations imply

$$
(\hat{R}_\lambda e, h) = \int_{-\infty}^{+\infty} \left( \frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) d(G(t)e, h)_E + (Qe, h)_E,
$$

hence,

$$
(V(\lambda)e, h) = \int_{-\infty}^{+\infty} \left( \frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) d(G(t)e, h)_E + (Qe, h)_E
$$

Next, we show that $(\mathbb{B} + iI) \hat{R}_\perp Ke = Ke$, for all $e \in E$, where $\mathbb{B}$ is the strong self-adjoint bi-extension defined by (98). By Theorem 7, the equation $(\mathbb{B} - \lambda I)x = f$ has a unique solution $x$ for any

$$
f \in \mathcal{K} \left[ \mathcal{C}_1^{-1} \left( S - \frac{i}{2} P_{\mathcal{P}_1} + \frac{i}{2} P_{\mathcal{P}_2} \right) \right] + E_\infty.
$$

We will now show that in fact

$$
\mathcal{K}(K) = \mathcal{K} \left[ \mathcal{C}_1^{-1} \left( S - \frac{i}{2} P_{\mathcal{P}_1} + \frac{i}{2} P_{\mathcal{P}_2} \right) \right] + E_\infty.
$$

If $\varphi_N \in N$, then

$$
\left( S - \frac{i}{2} P_{\mathcal{P}_1} + \frac{i}{2} P_{\mathcal{P}_2} \right) (\varphi_N - V\varphi_N) = \mathcal{C}_1(R_0^* + P_1^*) \tilde{R}_1^{-1} A^*(\varphi_N - V\varphi_N).
$$
Using (89) we can conclude that \( \tilde{R}^{-1}(I - V)N = F_\infty \), and hence

\[
\Re \left[ R_1^{-1} \left( S - \frac{i}{2} P^+_{\mathfrak{M}'} + \frac{i}{2} P^+_{\mathfrak{M}''} \right) \right] (I - V)N = (P^+_0 + R_0^*)F_\infty.
\]

Letting \( P^+ = P^+_{\mathfrak{M}'} + P^+_{\mathfrak{M}''} \), we have

\[
P^+ \left( S - \frac{i}{2} P^+_{\mathfrak{M}'} + \frac{i}{2} P^+_{\mathfrak{M}''} \right) (I + V)\varphi = 0, \varphi \in \mathfrak{M}.
\]

Therefore,

\[
E_\infty + \Re \left[ \tilde{R}^{-1} \left( S - \frac{i}{2} P^+_{\mathfrak{M}'} + \frac{i}{2} P^+_{\mathfrak{M}''} \right) \right] \Re(K).
\]

Since \( \hat{R}_\lambda = (B - \lambda I)^{-1} \), the above calculations imply

(106) \( (B - \lambda I)^{-1}Ke = \hat{R}_\lambda Ke \),

for all \( e \in E \). For \( \text{Im} \lambda \neq 0 \) we have that \( \hat{R}_\lambda KE = \mathfrak{N}_\lambda \) is the defect space of the operator \( A \). Therefore \( (B + iI)\hat{R}_{\pm i}Ke = Ke \) and \( \hat{R}_{\pm i}KE = \mathfrak{N}_{\pm i} \).

Taking into account (105) we get

\[
V(-i) = \int_{-\infty}^{+\infty} \left( \frac{1}{t + i} - \frac{t}{t^2 + 1} \right) dG(t) + Q
\]

(107)

\[
= -i \int_{-\infty}^{+\infty} \frac{dG(t)}{1 + t^2} + Q
\]

\[
= -iB + Q.
\]

Therefore,

(108) \( iV(-i)J + I = BJ + iQJ + I \).

The operator \( iV(-i)J + I \) is invertible and so is the right hand side of (108). Since \( I + BJ + iQJ = J(I + JB + iJQ)J \), where \( J \) is a unitary self-adjoint operator in the space \( E \), 0 is a regular point for the operator \( I + BJ + iJQ \). At the same time 0 is a regular point for the operators \( I + JB - iJQ = (BJ + iQJ + I)^* \) and \( I + BJ - iQJ = (I + JB + iJQ)^* \).

Let

(109)

\[
Z = (I + BJ - iQJ)^{-1}, \quad Z \in [E, E],
\]

\[
Z^* = (I + JB + iJQ)^{-1}, \quad Z^* \in [E, E],
\]
and let $\Gamma = (I + JB + iJQ)^{-1}$. Clearly $\text{Ker}\Gamma = 0$. We will show that for any $f \in E$, the equation

\[(110) \quad (A + iI)g = Kf,\]

has a unique solution $g = \hat{R}_{-i}K\Gamma f$, where $\hat{R}_{-i} = (\mathbb{B} + iI)^{-1}$ and $A = \mathbb{B} + i\mathbb{C}$. Moreover,

$\text{A}\hat{R}_{-i}K\Gamma f = \mathbb{B} \hat{R}_{-i}K\Gamma f + iKJK^*\hat{R}_{-i}K\Gamma f, \ f \in E.$

As shown above (see also [2])

\[K^* \hat{R}_{-i} \Gamma f = V(-i) \Gamma f = (Q - iB) \Gamma f,\]

\[iKJK^* \hat{R}_{-i}K\Gamma f = K(JB + iJQ) \Gamma f\]

\[= K(I + JB + iJQ)(I + JB + iJQ)^{-1}f - K\Gamma f\]

\[= Kf - K\Gamma f, \ f \in E.\]

Also,

\[(A + iI)\hat{R}_{-i}K\Gamma f = (\mathbb{B} + iI)\hat{R}_{-i}K\Gamma f + iKJK^*\hat{R}_{-i}K\Gamma f\]

\[= Kf, \ f \in E.\]

If there exists a $g \in \mathfrak{H}_i$ such that $\mathbb{A}g = -ig$, then $g \in \mathfrak{H}_{-i}$. Since $\mathfrak{H}(\Gamma) = E$, we find that $\hat{R}_{-i}K\Gamma E = \mathfrak{H}_{-i}$. Therefore $g = \hat{R}_{-i}K\Gamma e, e \in E$, and $(A + iI)\hat{R}_{-i}K\Gamma e = 0, Ke = 0, e = 0$, and $g = 0$. It follows that the equation $(A + iI)g = Kf$ has a unique solution given by $g = \hat{R}_{-i}K\Gamma f$ and $(A + iI)^{-1}KE = \mathfrak{H}_{-i}$.

Similarly, 0 is the regular point for the operator $I + JB - iJQ$ in $E$. Let

\[(111) \quad \Gamma_1 = (I + JB - iJQ)^{-1}.\]

In the same way as above, we can show that the equation $(\mathbb{A}^* - iI)gKf, f \in E$, has a unique solution of the form $g = \hat{R}_iK\Gamma_1 f$ and $(\mathbb{A}^* - iI)^{-1}KE = \mathfrak{H}_i$.

If $f_i \in \mathfrak{H}_i$, then $f_i = f_A + f_{\mathfrak{M}}$, where $f_A \in \mathcal{D}(A), f_{\mathfrak{M}} \in \mathfrak{M} = \mathfrak{N}'_i \oplus \mathfrak{N}'_{-i} \oplus \mathfrak{N}$. Therefore,

$A^* f_i = P A f_A + A^* f_{\mathfrak{M}} = iP f_i,$

$A^* f_{\mathfrak{M}} = iP f_i - PA f_A,$

and

\[(A + iI) f_i = (A + iI) f_A + iP f_i - PA f_A + i f_{\mathfrak{M}}\]

\[+ \mathcal{R}_1^{-1} \left( S - \frac{i}{2} P^*_i \mathfrak{N}'_{-i} + P^*_i \mathfrak{N}'_i \right) f_{\mathfrak{M}} + iKJK^* f_i,\]

\[= (I - P)(A + iI) f_A + i(P - I) f_i \in E \subset \mathfrak{H}(K).\]
This implies that 
\[(A + iI)f_i - 2if_i = (A + iI)f_i.\]
That is \[2if_i = (A + iI)(f_i - f_{-i}), \quad (f_{-i} \in \mathfrak{N}_{-i}).\] Hence \((A + iI)\mathfrak{H}^+ \subset \mathfrak{N}_i\). Since 
\[(A + iI)\mathcal{D}(A) = (A + iI)\mathcal{D}(A),\]
and \((A+iI)\mathcal{D}(A) \oplus \mathfrak{N}_i = \mathfrak{H}\), we have \((A+iI)\mathfrak{H}^+ \subset \mathfrak{H}\). Similarly, \((A^* - iI)\mathfrak{H}^+ \subset \mathfrak{H}\). Therefore we can conclude that the operators \((A + iI)^{-1}\) and \((A^* - iI)^{-1}\) are \((-\cdot, \cdot)-continuous\) (see [25]). Let
\[(112)\]
\[
\mathcal{D}(T) = (A + iI)^{-1}\mathfrak{H},
\]
\[
\mathcal{D}(T_1) = (A^* - iI)^{-1}\mathfrak{H}.
\]
It is easy to see that \(\mathcal{D}(T)\) and \(\mathcal{D}(T_1)\) are dense in \(\mathfrak{H}\) and that the operators \((A + iI)^{-1}\big|_{\mathfrak{H}}\) and \((A^* - iI)^{-1}\big|_{\mathfrak{H}}\) are \((\cdot, \cdot)-continuous\).

Let us define
\[(113)\]
\[
T = A\bigg|_{\mathcal{D}(T)},
\]
\[
T_1 = A^*\bigg|_{\mathcal{D}(T_1)}.
\]
The points \((i)\) and \((-i)\) are regular points for the operators \(T\) and \(T_1\) respectively. This implies that \(T_1 = T^*\).

Since \(T\) and \(T^*\) are quasi-kernels of operators \(A\) and \(A^*\) respectively, and \(\text{Re}A = \mathbb{B}\) is a strong self-adjoint bi-extension of the operator \(A\) we find that \(T \in \Lambda_A\) (the fact that \(PT\) and \(PT^*\) are closed follows from the \((+, \cdot)-continuity\) of \(T\) and \(T^*\)).

**Step 5.** Let us construct a linear stationary conservative dynamical system \(\theta\). Let \(K \in [E, \mathfrak{N}_-]\) be the operator defined in the Step 4. It is easy to see that
\[
\frac{1}{2i}(A - A^*) = KJK^*.
\]
Therefore,
\[
\theta = \begin{pmatrix}
\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- \\
A & K \\
J & E
\end{pmatrix}
\]
is a l.s.c.d.s. In particular, \(\theta\) is a scattering system if \(J = I\). Since \(V_\theta(z)\) is a linear-fractional transformation of \(W_\theta(z)\) then \(V_\theta(z) = V(z)\) whenever \(z\) is in some neighborhood \(G_{-i}\) of the point \((-i)\). This completes the proof of the theorem.

**Remark.** It can be seen that when \(J = I\) the invertibility condition for \(I + iV(\lambda)J\) is satisfied automatically.
Theorem 10. Let an operator-valued function $V(z)$ belong to the class $N(R)$. Then $V(z)$ can be realized by the scattering ($J = I$) system (dissipative operator colligation) $\theta$ of the form (30).

The following theorem deals with the realization of two realizable operator-valued $R$-functions differing from each other only by the constant terms in the representation (48).

Theorem 11. Let the operator-valued functions

$$V_1(\lambda) = Q_1 + \int_{-\infty}^{+\infty} \left( \frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) dG(t)$$

and

$$V_2(\lambda) = Q_2 + \int_{-\infty}^{+\infty} \left( \frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) dG(t)$$

belong to the class $N(R)$. Then they can be realized by systems

$$\theta_1 = \begin{pmatrix} A_1 & K_1 & J \\ \mathfrak{H}_+ & \mathfrak{H} & \mathfrak{H}_- \\ \mathbb{E} \end{pmatrix} \quad (A_1 \supset T_1)$$

and

$$\theta_2 = \begin{pmatrix} A_2 & K_2 & J \\ \mathfrak{H}_+ & \mathfrak{H} & \mathfrak{H}_- \\ \mathbb{E} \end{pmatrix} \quad (A_2 \supset T_2)$$

respectively, so that the operators $T_1$ and $T_2$ acting on the Hilbert space $\mathfrak{H}$ are both extensions of the Hermitian operator $A$ defined in this Hilbert space.

Proof. Applying Theorem 9 to the function $V_1(\lambda)$, we obtain a l.s.c.d.s. $\theta_1$ of the type (116). The corresponding Hermitian operator $A_1$ constructed in the Steps 1 and 2 of the proof of Theorem 9 satisfies the formulas (72) and (73). The construction of $A_1$ doesn’t involve the operator $Q_1$ from (114). It is easy to see that the corresponding rigged Hilbert space $\mathfrak{H}_+^{(1)} \subset \mathfrak{H}_+ \subset \mathfrak{H}_-^{(1)}$ was built without the use of the operator $Q_1$ too.

Similarly, if we apply Theorem 9 to the function $V_2(\lambda)$ we get the corresponding Hermitian operator $A_2 = A_1$ and the same rigged Hilbert space. This occurs because the operator-functions $V_1(\lambda)$ and $V_2(\lambda)$ differ from each other only by the constant terms $Q_1$ and $Q_2$. Setting $A = A_1 = A_2$, we can conclude that $T_1$ and $T_2$ are both extensions of the Hermitian operator $A$.

A closed Hermitian operator $A$ is called a prime operator [25] if there exists no reducing invariant subspace on which it induces a self-adjoint operator.
**Definition.** A l.s.c.d.s. $\theta$ of the form (30) is said to be a **prime system** if its Hermitian operator $A$ is a prime operator.

**Theorem 12.** Let the operator-valued function $V(z)$ belong to the class $N(R)$. Then it can be realized by the prime system $\theta$ of the form (30) with a preassigned direction operator $J$ for which $I + iV(-i)J$ is invertible.

**Proof.** Theorem 9 provides us with a possibility of realization for a given operator-valued function $V(z)$ from the class $N(R)$. Let us assume that its Hermitian operator $A$ has a reducing invariant subspace $H_1 \subset H$ on which it generates the self-adjoint operator $A_1$. Then we can write the following ($\cdot$)-orthogonal decomposition

\[
H = H_0 + H_1, \quad A = A_0 \oplus A_1,
\]

where $A_0$ is an operator induced by $A$ on $H_0$.

Now let us consider an operator $T \supset A$ as in the definition of the system $\theta$. We have

\[
T = T_0 \oplus A_1,
\]

where $T_0 \supset A_0$. Indeed, since $A_1$ is a self-adjoint operator it can not be extended any further. Clearly, $\mathcal{D}(A_1) = H_1$. Similarly,

\[
T^* = T_0^* \oplus A_1,
\]

where $T_0^* \supset A_0$. Furthermore,

\[
\mathcal{H}_+ = \mathcal{H}_0^0 \oplus \mathcal{H}_1^1 = \mathcal{D}(A_0^*) \oplus \mathcal{D}(A_1).
\]

We now show that the same holds in the (+)-orthogonality sense. Indeed, if $f_0 \in \mathcal{H}_0^0$, $f_1 \in \mathcal{H}_1^1 = \mathcal{D}(A_1)$ then

\[
(f_0, f_1)_+ = (f_0, f_1) + (A^* f_0, A^* f_1)
= (f_0, f_1) + (A_0^* f_0, A_1 f_1)
= 0 + 0 = 0.
\]

Consequently, we have

\[
\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- = \mathcal{H}_0^0 \oplus \mathcal{H}_1^1 \subset \mathcal{H}_0^0 \oplus \mathcal{H}_1^1 \subset \mathcal{H}_0^0 \oplus \mathcal{H}_1^1 = \mathcal{H}_0^0 \oplus \mathcal{D}(A_1) \subset \mathcal{H}_0^0 \oplus \mathcal{D}(A_1) \subset H = \mathcal{H}_0^0 \oplus \mathcal{H}_1^1.
\]

Similarly, we obtain $A = A_0 \oplus A_1$ and $A^* = A_0 \oplus A_1$. Therefore,

\[
\frac{A - A^*}{2i} = \frac{(A_0 \oplus A_1) - (A_0^* \oplus A_1)}{2i}
= \frac{A_0 - A_0^*}{2i} \oplus \frac{A_1 - A_1}{2i}
= \frac{A_0 - A_0^*}{2i} \oplus O,
\]
where $O$ is the zero operator. This implies that

\[ KJK^* = K_0JK_0^* \oplus O. \]

Let $P_+^0$ be an orthoprojection operator of $\mathcal{H}_+$ onto $\mathcal{H}_+^0$ and set $K = K_0$. Now $K^* = K_0^*P_+^0$, since for all $f \in E$, $g \in \mathcal{H}_+$ we have:

\[
(Kf, g) = (K_0f, g) = (K_0f, g_0 + g_1) = (K_0^*f, g_0) + (K_0^*f, g_1) \\
= (K_0f, g_0) = (f, K_0^*g_0) = (f, K_0^*P_+^0g).
\]

Next, consider $e \in E$ and $x = x^0 + x^1$ in $\mathcal{H}_+$ such that

\[
(\mathbb{A} - \lambda I)P_+^0x = Ke.
\]

Then

\[
(\mathbb{A}_0 \oplus A_1 - \lambda I)P_+^0x = K_0e, \\
\mathbb{A}_0x^0 - \lambda x^0 = K_0e, \\
(\mathbb{A} - \lambda I)x^0 = K_0e,
\]

\[
x^0 = (\mathbb{A}_0 - \lambda I)^{-1}K_0e.
\]

On the other hand, $x^0 = (\mathbb{A} - \lambda I)^{-1}Ke$. Therefore

\[
(\mathbb{A} - \lambda I)^{-1}Ke = (\mathbb{A}_0 - \lambda I)^{-1}K_0e,
\]

and

\[
K^*(\mathbb{A} - \lambda I)^{-1}Ke = K_0^*(\mathbb{A}_0 - \lambda I)^{-1}K_0e.
\]

This means that the transfer operator-functions of our system $\theta$ and of the system

\[
\theta_0 = \left( \begin{array}{cc}
\mathbb{A}_0 & K_0 \\
\mathcal{H}_+ & J \\
E & H
\end{array} \right)
\]

coincide. This proves the statement of the theorem.

4. Example

Let

\[
Tx = \frac{1}{i} \frac{dx}{dt},
\]

with

\[
\mathcal{D}(T) = \left\{ x(t) : x'(t) \in L^2_{[0,l]}, x(0) = 0 \right\},
\]
be a differential operator in $\mathcal{H} = L^2_{[0,l]}$ ($l > 0$). Obviously,

$$T^* x = \frac{1}{i} \frac{dx}{dt},$$

with

$$\mathcal{D}(T^*) = \left\{ x(t) : x'(t) \in L^2_{[0,l]}, x(l) = 0 \right\},$$

is the adjoint operator of $T$. Consider the Hermitian operator $A$ (see also [1]) defined by

$$Ax = \frac{1}{i} \frac{dx}{dt},$$

$$\mathcal{D}(A) = \left\{ x(t) : x'(t) \in L^2_{[0,l]}, x(0) = x(l) = 0 \right\},$$

where its adjoint $A^*$ is given by

$$A^* x = \frac{1}{i} \frac{dx}{dt},$$

$$\mathcal{D}(A^*) = \left\{ x(t) : x'(t) \in L^2_{[0,l]} \right\}.$$

Then $\mathcal{H}_+ = \mathcal{D}(A^*) = W^1_2$ is a Sobolev space with scalar product

$$(x, y)_+ = \int_0^l x(t) \overline{y(t)} \, dt + \int_0^l x'(t) \overline{y'(t)} \, dt.$$

We construct the rigged Hilbert space [9]

$$W^1_2 \subset L^2_{[0,l]} \subset (W^1_2)^{-},$$

and consider the operators

$$\mathbb{A} x = \frac{1}{i} \frac{dx}{dt} + ix(0) [\delta(x - l) - \delta(x)],$$

$$\mathbb{A}^* x = \frac{1}{i} \frac{dx}{dt} + ix(l) [\delta(x - l) - \delta(x)],$$

where $x(t) \in W^1_2$, $\delta(x)$, $\delta(x - l)$ are delta-functions in $(W^1_2)^{-}$. It is easy to see that

$$\mathbb{A} \supset T \supset A, \quad \mathbb{A}^* \supset T^* \supset A,$$

and

$$\theta = \begin{pmatrix} \frac{1}{i} \frac{dx}{dt} + ix(0)[\delta(x - l) - \delta(x)] & K & -1 \\ W^2_1 \subset L^2_{[0,l]} \subset (W^1_2)^{-} & \mathbb{C}^1 \\
90 \end{pmatrix} (J = -1)$$
is a Brodskii-Livšic rigged operator colligation where

\[ Kc = c \cdot \frac{1}{\sqrt{2}} [\delta(x - l) - \delta(x)], \quad (c \in \mathbb{C}^1) \]

\[ K^*x = \left( x, \frac{1}{\sqrt{2}} [\delta(x - l) - \delta(x)] \right) = \frac{1}{\sqrt{2}} [x(l) - x(0)], \]

for \( x(t) \in W_2^1 \). Also

\[ \frac{A - A^*}{2i} = - \left( \frac{1}{\sqrt{2}} [\delta(x - l) - \delta(x)] \right) \frac{1}{\sqrt{2}} [\delta(x - l) - \delta(x)]. \]

The characteristic function of this colligation is

\[ W_\theta(\lambda) = I - 2iK^*(A - \lambda I)^{-1}KJ = e^{i\lambda l}. \]

Consider the following \( R \)-function (hyperbolic tangent)

\[ V(\lambda) = -i \tanh \left( \frac{i}{2} \lambda l \right). \]

Obviously this function can be realized as follows

\[ V(\lambda) = -i \tanh \left( \frac{i}{2} \lambda l \right) = -i \frac{e^{\frac{i}{2} \lambda l} - e^{-\frac{i}{2} \lambda l}}{e^{\frac{i}{2} \lambda l} + e^{-\frac{i}{2} \lambda l}} = -i \frac{e^{i\lambda l} - 1}{e^{i\lambda l} + 1} = i [W_\theta(\lambda) + I]^{-1} [W_\theta(\lambda) - I] J. \quad (J = -1) \]

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Department of Mathematics
Troy State University
Troy, AL 36082
E-mail address: sbelyi@trojan.troyst.edu

Department of Mathematics
University of Missouri-Columbia
Columbia, MO 65211
E-mail address: tsekanov@math.missouri.edu

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