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REALIZATION THEOREMS FOR OPERATOR-VALUED R-FUNCTIONS

S.V. Belyi and E.R. Tsekanovskii

Dedicated to the memory of Professor Israel Glazman

In this paper we consider realization problems for operator-valued R-functions acting on a Hilbert space E ($\dim E < \infty$) as linear-fractional transformations of the transfer operator-valued functions (characteristic functions) of linear stationary conservative dynamic systems (Brodskiĭ-Livšic rigged operator colligations). We give complete proofs of both the direct and inverse realization theorems announced in [6], [7].

1. Introduction

Realization theory of different classes of operator-valued (matrix-valued) functions as transfer operator-functions of linear systems plays an important role in modern operator and systems theory. Almost all realizations in the modern theory of non-selfadjoint operators and its applications deal with systems (operator colligations) in which the main operators are *bounded* linear operators [8], [10-14], [17], [21]. The realization with an *unbounded* operator as a main operator in a corresponding system has not been investigated thoroughly because of a number of essential difficulties usually related to unbounded non-selfadjoint operators.

We consider realization problems for operator-valued R-functions acting on a finite dimensional Hilbert space E as linear-fractional transformations of the transfer operator-functions of linear stationary conservative dynamic systems (l.s.c.d.s.) θ of the form

$$\begin{cases} (\mathbb{A} - zI)x = KJ\varphi_- \\ \varphi_+ = \varphi_- - 2iK^*x \end{cases} \qquad (\mathrm{Im}\ \mathbb{A} = KJK^*),$$

or

$$\theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & E \end{pmatrix}.$$

In the system θ above $\mathbb{A}$ is a bounded linear operator, acting from $\mathfrak{H}_+$ into $\mathfrak{H}_-$, where $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$ is a rigged Hilbert space, $\mathbb{A} \supset T \supset A$, $\mathbb{A}^* \supset T^* \supset A$, A is a Hermitian operator in $\mathfrak{H}$, T is a non-Hermitian operator in $\mathfrak{H}$, K is a linear bounded operator from E into $\mathfrak{H}_-$, $J = J^* = J^{-1}$, $\varphi_\pm \in E$, φ_- is an input vector, φ_+ is an output vector, and $x \in \mathfrak{H}_+$ is a vector of the inner state of the system θ. The operator-valued function

$$W_\theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ \qquad (\varphi_+ = W_\theta(z)\varphi_-),$$

is the transfer operator-function of the system θ.

We establish criteria for a given operator-valued R-function $V(z)$ to be realized in the form

$$V(z) = i[W_\theta(z) + I]^{-1}[W_\theta(z) - I]J.$$

It is shown that an operator-valued R-function

$$V(z) = Q + F \cdot z + \int_{-\infty}^{+\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) dG(t),$$

acting on a Hilbert space E ($\dim E < \infty$) with some invertibility condition can be realized if and only if

$$F = 0 \quad \text{and} \quad Qe = \int\limits_{-\infty}^{+\infty} \frac{t}{1+t^2}\, dG(t)e,$$

for all $e \in E$ such that

$$\int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty.$$

Moreover, if two realizable operator-valued R-functions are different only by a constant term then they can be realized by two systems θ_1 and θ_2 with corresponding non-selfadjoint operators that have the same Hermitian part A.

The rigged operator colligation θ mentioned above is exactly an unbounded version of the well known Brodskiĭ-Livšic bounded operator colligation α of the form [11]

$$\alpha = \begin{pmatrix} T & K & J \\ \mathfrak{H} & & E \end{pmatrix} \qquad (\mathrm{Im}\ T = KJK^*),$$

with a bounded linear operator T in $\mathfrak{H}$ (and without rigged Hilbert spaces).

To prove the direct and inverse realization theorems for operator-valued R-functions we build a functional model which generally speaking is an unbounded version of the

Brodskiĭ-Livšic model with diagonal real part. This model for bounded linear operators was constructed in [11].

When this paper was submitted for publication, an article by D. Arov and M. Nudelman [5] appeared considering realization problem for another class of operator-valued functions (contractive) but not in terms of rigged operator colligations. At the end of this paper there is an example showing how a given R-function can be realized by a rigged operator colligation.

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2. Preliminaries

In this section we recall some basic definitions and results that will be used in the proof of the realization theorem.

The Rigged Hilbert Spaces. Let $\mathfrak{H}$ denote a Hilbert space with inner product (x, y) and let A be a closed linear Hermitian operator, i.e. $(Ax, y) = (x, Ay)$ $(\forall x, y \in \mathfrak{D}(A))$, acting in the Hilbert space $\mathfrak{H}$ with generally speaking, non-dense domain $\mathfrak{D}(A)$. Let $\mathfrak{H}_0 = \overline{\mathfrak{D}(A)}$ and A^* be the adjoint to the operator A (we consider A acting from $\mathfrak{H}_0$ into $\mathfrak{H}$).

Now we are going to equip $\mathfrak{H}$ with spaces $\mathfrak{H}_+$ and $\mathfrak{H}_-$ called, respectively, spaces with positive and negative norms [9]. We denote $\mathfrak{H}_+ = \mathfrak{D}(A^*)$ $((\overline{\mathfrak{D}(A^*)} = \mathfrak{H})$ with inner product

$$(f, g)_+ = (f, g) + (A^*f, A^*g) \qquad (f, g \in \mathfrak{H}_+), \tag{1}$$

and then construct the *rigged* Hilbert space $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$. Here $\mathfrak{H}_-$ is the space of all linear functionals over $\mathfrak{H}_+$ that are continuous with respect to $\|\cdot\|_+$. The norms of these spaces are connected by the relations $\|x\| \le \|x\|_+$ $(x \in \mathfrak{H}_+)$, and $\|x\|_- \le \|x\|$ $(x \in \mathfrak{H})$. It is well known that there exists an isometric operator $\mathcal{R}$ which maps $\mathfrak{H}_-$ onto $\mathfrak{H}_+$ such that

$$\begin{aligned} (x, y)_- = (x, \mathcal{R}y) = (\mathcal{R}x, y) = (\mathcal{R}x, \mathcal{R}y)_+ \qquad (x, y \in \mathfrak{H}_-), \\ (u, v)_+ = (u, \mathcal{R}^{-1}v) = (\mathcal{R}^{-1}u, v) = (\mathcal{R}^{-1}u, \mathcal{R}^{-1}v)_- \qquad (u, v \in \mathfrak{H}_+). \end{aligned} \tag{2}$$

The operator $\mathcal{R}$ will be called the Riesz-Berezanskii operator. In what follows we use symbols $(+)$, $(\cdot)$, and $(-)$ to indicate the norms $\|\cdot\|_+$, $\|\cdot\|$, and $\|\cdot\|_-$ by which geometrical and topological concepts are defined in $\mathfrak{H}_+$, $\mathfrak{H}$, and $\mathfrak{H}_-$.

Analogues of von Neumann's formulae. It is easy to see that for a Hermitian operator A in the above settings $\mathfrak{D}(A) \subset \mathfrak{D}(A^*)(= \mathfrak{H}_+)$ and $A^*y = PAy$ $(\forall y \in \mathfrak{D}(A))$, where P is an orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_0$. We put

$$\mathfrak{L} := \mathfrak{H} \ominus \mathfrak{H}_0 \quad \mathfrak{M}_\lambda := (A - \lambda I)\mathfrak{D}(A) \quad \mathfrak{N}_\lambda := (\mathfrak{M}_{\bar\lambda})^\perp \tag{3}$$

The subspace $\mathfrak{N}_\lambda$ is called a *defect subspace* of A for the point $\bar{\lambda}$. The cardinal number $\dim\mathfrak{N}_\lambda$ remains constant when λ is in the upper half-plane. Similarly, the number $\dim\mathfrak{N}_\lambda$ remains constant when λ is in the lower half-plane. The numbers $\dim\mathfrak{N}_\lambda$ and $\dim\mathfrak{N}_{\bar{\lambda}}$ ($\mathrm{Im}\lambda < 0$) are called the *defect numbers* or *deficiency indices* of operator A [1]. The subspace $\mathfrak{N}_\lambda$ which lies in $\mathfrak{H}_+$ is the set of solutions of the equation $A^*g = \lambda Pg$.

Let now P_λ be the orthogonal projection onto $\mathfrak{N}_\lambda$, set

$$\mathfrak{B}_\lambda = P_\lambda \mathfrak{L}, \qquad \mathfrak{N}'_\lambda = \mathfrak{N}_\lambda \ominus \overline{\mathfrak{B}_\lambda} \tag{4}$$

It is easy to see that $\mathfrak{N}'_\lambda = \mathfrak{N}_\lambda \cap \mathfrak{H}_0$ and $\mathfrak{N}'_\lambda$ is the set of solutions of the equation $A^*g = \lambda g$ (see [25]), when $A^* : \mathfrak{H} \to \mathfrak{H}_0$ is the adjoint operator to A.

The subspace $\mathfrak{N}'_\lambda$ is the defect subspace of the densely defined Hermitian operator PA on $\mathfrak{H}_0$ ([22]). The numbers $\dim\mathfrak{N}'_\lambda$ and $\dim\mathfrak{N}'_{\bar{\lambda}}$ ($\mathrm{Im}\lambda < 0$) are called *semi-defect numbers* or the *semi-deficiency indices* of the operator A [16]. The von Neumann formula

$$\mathfrak{H}_+ = \mathfrak{D}(A^*) = \mathfrak{D}(A) + \mathfrak{N}_\lambda + \mathfrak{N}_{\bar{\lambda}}, \qquad (\mathrm{Im}\lambda \neq 0), \tag{5}$$

holds, but this decomposition is not direct for a non-densely defined operator A. There exists a generalization of von Neumann's formula [3], [24] to the case of a non-densely defined Hermitian operator (direct decomposition).

We call an operator A *regular*, if PA is a closed operator in $\mathfrak{H}_0$. For a regular operator A we have

$$\mathfrak{H}_+ = \mathfrak{D}(A) + \mathfrak{N}'_\lambda + \mathfrak{N}'_{\bar{\lambda}} + \mathfrak{N}, \qquad (\mathrm{Im}\lambda \neq 0) \tag{6}$$

where $\mathfrak{N} := \mathcal{R}\mathfrak{L}$. This is a generalization of von Neumann's formula. For $\lambda = \pm i$ we obtain the (+)-orthogonal decomposition

$$\mathfrak{H}_+ = \mathfrak{D}(A) \oplus \mathfrak{N}'_i \oplus \mathfrak{N}'_{-i} \oplus \mathfrak{N}. \tag{7}$$

Let $\tilde{A}$ be a closed Hermitian extension of the operator A. Then $\mathfrak{D}(\tilde{A}) \subset \mathfrak{H}_+$ and $P\tilde{A}x = A^*x$ ($\forall x \in \mathfrak{D}(\tilde{A})$). According to [25] a closed Hermitian extension $\tilde{A}$ is said to be *regular* if $\mathfrak{D}(\tilde{A})$ is (+)-closed. According to the theory of extensions of closed Hermitian operators A with non-dense domain [16], an operator U ($\mathfrak{D}(U) \subseteq \mathfrak{N}_i$, $\mathfrak{R}(U) \subseteq \mathfrak{N}_{-i}$) is called an *admissible operator* if $(U - I)f_i \in \mathfrak{D}(A)$ ($f_i \in \mathfrak{D}(U)$) only for $f_i = 0$. Then (see [4]) any symmetric extension $\tilde{A}$ of the non-densely defined closed Hermitian operator A, is defined by an isometric admissible operator U, $\mathfrak{D}(U) \subseteq \mathfrak{N}_i$, $\mathfrak{R}(U) \subseteq \mathfrak{N}_{-i}$ by the formula

$$\tilde{A}f_{\tilde{A}} = Af_A + (-if_i - iUf_i), \quad f_A \in \mathfrak{D}(A) \tag{8}$$

where $\mathfrak{D}(\tilde{A}) = \mathfrak{D}(A) \dot{+} (U-I)\mathfrak{D}(U)$. The operator $\tilde{A}$ is self-adjoint if and only if $\mathfrak{D}(U) = \mathfrak{N}_i$ and $\mathfrak{R}(U) = \mathfrak{N}_{-i}$.

Let us denote now by $P^+_{\mathfrak{N}}$ the orthogonal projection operator in $\mathfrak{H}_+$ onto $\mathfrak{N}$. We introduce a new inner product $(\cdot,\cdot)_1$ defined by

$$(f,g)_1 = (f,g)_+ + (P^+_{\mathfrak{N}} f, P^+_{\mathfrak{N}} g)_+ \tag{9}$$

for all $f,g \in \mathfrak{H}_+$. The obvious inequality

$$\|f\|^2_+ \le \|f\|^2_1 \le 2\|f\|^2_+$$

shows that the norms $\|\cdot\|_+$ and $\|\cdot\|_1$ are topologically equivalent. It is easy to see that the spaces $\mathfrak{D}(A)$, $\mathfrak{N}'_i$, $\mathfrak{N}'_{-i}$, $\mathfrak{N}$ are (1)-orthogonal. We write $\mathfrak{M}_1$ for the Hilbert space $\mathfrak{M} = \mathfrak{N}'_i \oplus \mathfrak{N}'_{-i} \oplus \mathfrak{N}$ with inner product $(f,g)_1$. We denote by $\mathfrak{H}_{+1}$ the space $\mathfrak{H}_+$ with norm $\|\cdot\|_1$, and by $\mathcal{R}_1$ the corresponding Riesz-Berezanskii operator related to the rigged Hilbert space $\mathfrak{H}_{+1} \subset \mathfrak{H} \subset \mathfrak{H}_{-1}$. The following theorem gives a characterization of the regular extensions for a regular closed Hermitian operator A (see [4]).

Theorem 1. *I. For each closed Hermitian extension $\tilde{A}$ of a regular operator A there exists a (1)-isometric operator $V = V(\tilde{A})$ on $\mathfrak{M}_1$ with the properties: a) $\mathfrak{D}(V)$ is (+)-closed and belongs to $\mathfrak{N} \oplus \mathfrak{N}'_i$, $\mathfrak{R}(V) \subset \mathfrak{N} \oplus \mathfrak{N}'_{-i}$; b) $Vh = h$ only for $h = 0$, and $\mathfrak{D}(\tilde{A}) = \mathfrak{D}(A) \oplus (I+V)\mathfrak{D}(V)$.*

Conversely, for each (1)-isometric operator V with the properties a) and b) there exists a closed Hermitian extension $\tilde{A}$ in the sense indicated.

II. The extension $\tilde{A}$ is regular if and only if the manifold $\mathfrak{R}(I+V)$ is (1)-closed.

III. The operator $\tilde{A}$ is self-adjoint if and only if $\mathfrak{D}(V) = \mathfrak{N} \oplus \mathfrak{N}'_i$, $\mathfrak{R}(V) = \mathfrak{N} \oplus \mathfrak{N}'_{-i}$.

The following theorem can be found in [16].

Theorem 2. *Let $\tilde{A}$ be a regular self-adjoint extension of a regular Hermitian operator A, that is determined by an admissible operator U and let*

$$\hat{\mathfrak{N}}_i = \{f_i \in \mathfrak{N}_i, (U-I)f_i \in \mathfrak{H}_0\}. \tag{10}$$

Then

$$\mathfrak{H}_+ = \mathfrak{D}(\tilde{A}) \dot{+} (U+I)\hat{\mathfrak{N}}_i. \tag{11}$$

Bi-extensions. Denote by $[\mathfrak{H}_1, \mathfrak{H}_2]$ the set of all linear bounded operators acting from the Hilbert space $\mathfrak{H}_1$ into the Hilbert space $\mathfrak{H}_2$.

Definition. An operator $\mathbb{A} \in [\mathfrak{H}_+, \mathfrak{H}_-]$ is a *bi-extension* of A if both $\mathbb{A} \supset A$ and $\mathbb{A}^* \supset A$.

If $\mathbb{A} = \mathbb{A}^*$, then $\mathbb{A}$ is called a self-adjoint bi-extension of the operator A. We write $\mathfrak{S}(A)$ for the class of bi-extensions of A. This class is closed in the weak topology and is invariant under taking adjoints. The following theorem from [4], [25] gives a description of $\mathfrak{S}(A)$.

Theorem 3. *Every bi-extension $\mathbb{A}$ of a regular Hermitian operator A has the form:*

$$\mathbb{A} = AP^+_{\mathfrak{D}(A)} + [A^* + \mathcal{R}_1^{-1}(Q - \frac{i}{2}P^+_{\mathfrak{N}'_i} + \frac{i}{2}P^+_{\mathfrak{N}'_{-i}})]P^+_{\mathfrak{M}} \tag{12}$$

$$\mathbb{A}^* = AP^+_{\mathfrak{D}(A)} + [A^* + \mathcal{R}_1^{-1}(Q^* - \frac{i}{2}P^+_{\mathfrak{N}'_i} + \frac{i}{2}P^+_{\mathfrak{N}'_{-i}})]P^+_{\mathfrak{M}} \tag{13}$$

where Q is an arbitrary operator in $[\mathfrak{M}, \mathfrak{M}]$ and Q^* is its adjoint with respect to the (1)-metric.

Corollary 1. *Every self-adjoint bi-extension $\mathbb{A}$ of the regular Hermitian operator A is of the form:*

$$\mathbb{A} = AP^+_{\mathfrak{D}(A)} + [A^* + \mathcal{R}_1^{-1}(S - \frac{i}{2}P^+_{\mathfrak{N}'_i} + \frac{i}{2}P^+_{\mathfrak{N}'_{-i}})]P^+_{\mathfrak{M}}, \tag{14}$$

where S is an arbitrary (1)-self-adjoint operator in $[\mathfrak{M}, \mathfrak{M}]$.

Let $\mathbb{A}$ be a bi-extension of a Hermitian operator A. The operator $\hat{A}f = \mathbb{A}f$, $\mathfrak{D}(\hat{A}) = \{f \in \mathfrak{H}, \mathbb{A}f \in \mathfrak{H}\}$ is called the *quasi-kernel* of $\mathbb{A}$. If $\mathbb{A} = \mathbb{A}^*$ and $\hat{A}$ is a quasi-kernel of $\mathbb{A}$ such that $A \neq \hat{A}$, $\hat{A}^* = \hat{A}$ then $\mathbb{A}$ is said to be a *strong* self-adjoint bi-extension of A.

Classes Ω_A and Λ_A. $(*)$-extensions. Let A be a closed Hermitian operator.

Definition. We say that a closed densely defined linear operator T acting on the Hilbert space $\mathfrak{H}$ belongs to the class Ω_A if:

(1) $T \supset A$ and $T^* \supset A$;
(2) $(-i)$ is a regular point of T.[1]

It was mentioned in [4] that sets $\mathfrak{D}(T)$ and $\mathfrak{D}(T^*)$ are $(+)$-closed, the operators T and T^* are $(+, \cdot)$-bounded. The following theorem [25] is an analogue of von Neumann's formulae for the class Ω_A.

[1] The condition, that $(-i)$ is a regular point in the definition of the class Ω_A is not essential. It is sufficient to require the existence of some regular point for T.

Theorem 4. *I. To each operator of the class Ω_A there corresponds an operator M on the space $\mathfrak{M}_1$ with the following properties:*

(1) $\mathfrak{D}(M)=\mathfrak{N}'_i\oplus\mathfrak{N}$, *and* $\mathfrak{R}(M)=\mathfrak{N}'_{-i}\oplus\mathfrak{N}$*;*

(2) $Mx+x=0$ *only for* $x=0$*, and* $M^*x+x=0$ *only for* $x=0$*. Moreover, the following hold:*

$$\mathfrak{D}(T)=\mathfrak{D}(A)\oplus(M+I)(\mathfrak{N}'_i\oplus\mathfrak{N}), \tag{15}$$

$$\mathfrak{D}(T^*)=\mathfrak{D}(A)\oplus(M^*+I)(\mathfrak{N}'_{-i}\oplus\mathfrak{N}). \tag{16}$$

II. Conversely, for each pair of (1)-adjoint operators M and M^* in $[\mathfrak{M}_1,\mathfrak{M}_1]$ satisfying (1) and (2) above, formulas (15) and (16) give a corresponding operator T in the class Ω_A. Moreover, if $f=g+(M+I)\varphi$, $g\in\mathfrak{D}(A)$, and $\varphi\in\mathfrak{N}'_i\oplus\mathfrak{N}$ then

$$Tf=Ag+A^*(I+M)\varphi+i\mathcal{R}_1^{-1}P^+_{\mathfrak{N}}(I-M)\varphi\quad(f\in\mathfrak{D}(T)). \tag{17}$$

Similarly, if $f=g+(M^*+I)\psi$, $g\in\mathfrak{D}(A)$, and $\psi\in\mathfrak{N}'_{-i}\oplus\mathfrak{N}$, then

$$T^*f=Ag+A^*(I+M^*)\psi+i\mathcal{R}_1^{-1}P^+_{\mathfrak{N}}(M^*-I)\psi\quad(f\in\mathfrak{D}(T)), \tag{18}$$

Definition. An operator $\mathbb{A}$ in $[\mathfrak{H}_+,\mathfrak{H}_-]$ is called a $(*)$-extension of an operator T from the class Ω_A if both $\mathbb{A}\supset T$ and $\mathbb{A}^*\supset T^*$.

This $(*)$-extension is called *correct*, if an operator $\mathbb{A}_R=\frac{1}{2}(\mathbb{A}+\mathbb{A}^*)$ is a strong self-adjoint bi-extension of an operator A. It is easy to show that if $\mathbb{A}$ is a $(*)$-extension of T, then T and T^* are quasi-kernels of $\mathbb{A}$ and $\mathbb{A}^*$, respectively.

Definition. We say that the operator T of the class Ω_A belongs to the class Λ_A if

(1) T admits a correct $(*)$-extension;

(2) A is the maximal common Hermitian part of T and T^*.

Theorem 5. *Let an operator T belong to Ω_A and let M be an operator in $[\mathfrak{M},\mathfrak{M}]$ that is related to T by Theorem 4. Then T belongs to Λ_A if and only if there exists either* (1)*-isometric operator or a* $(\cdot)$*-isometric operator U in $[\mathfrak{N}'_i,\mathfrak{N}'_{-i}]$ such that*

$$\begin{cases}(U+I)\mathfrak{N}'_i+(M+I)(\mathfrak{N}'_i\oplus\mathfrak{N})=\mathfrak{M},\\(U+I)\mathfrak{N}'_i+(M+I)(\mathfrak{N}'_i\oplus\mathfrak{N})=\mathfrak{M}.\end{cases} \tag{19}$$

Corollary 2. *If a closed Hermitian operator A has finite and equal defect indices, then the class Ω_A coincides with the Λ_A.*

Extended Resolvents and Extended Spectral Functions of a Hermitian Operator. Let A be a closed Hermitian operator on $\mathfrak{H}$ and $\mathfrak{h}$ be a Hilbert space such that $\mathfrak{H}$ is a subspace of $\mathfrak{h}$. Let $\tilde{A}$ be a self-adjoint extension of A on $\mathfrak{h}$, and $\tilde{E}(t)$ be the spectral function of $\tilde{A}$. An operator function $R_\lambda = P_{\mathfrak{H}}(\tilde{A} - \lambda I)^{-1}|_{\mathfrak{H}}$ is called a *generalized resolvent* of A, and $E(t) = P_{\mathfrak{H}}\tilde{E}(t)|_{\mathfrak{H}}$ is the corresponding *generalized spectral function*. Here

$$
R_\lambda = \int\limits_{-\infty}^{\infty} \frac{dE(t)}{t-\lambda} \quad (\mathrm{Im}\lambda \neq 0). \tag{20}
$$

If $\mathfrak{h} = \mathfrak{H}$ then R_λ and $E(t)$ are called *canonical resolvent* and *canonical spectral function*, respectively. According to [19] we denote by $\hat{R}_\lambda$ the $(-,\cdot)$-continuous operator from $\mathfrak{H}_-$ into $\mathfrak{H}$ which is adjoint to $R_{\bar{\lambda}}$:

$$
(\hat{R}_\lambda f, g) = (f, R_{\bar{\lambda}g}) \quad (f \in \mathfrak{h}_-,\, g \in \mathfrak{H}). \tag{21}
$$

It follows that $\hat{R}_\lambda f = R_\lambda f$ for $f \in h$, so that $\hat{R}_\lambda$ is an extension of R_λ from $\mathfrak{H}$ to $\mathfrak{H}_-$ with respect to $(-,\cdot)$-continuity. The function $\hat{R}_\lambda$ of the parameter λ, $(\mathrm{Im}\lambda \neq 0)$ is called the *extended generalized (canonical) resolvent* of the operator A. We write $\aleph$ for the family of all finite intervals on the real axis. It is known [19] that if $\varDelta \in \aleph$ then $E(\varDelta)\mathfrak{H} \subset \mathfrak{H}_+$ and the operator $E(\varDelta)$ is $(\cdot,+)$-continuous. We denote by $\hat{E}(\varDelta)$ the $(-,\cdot)$-continuous operator from $\mathfrak{H}_-$ to $\mathfrak{H}$ that is adjoint to $E(\varDelta) \in [\mathfrak{H}, \mathfrak{H}_+]$. Similarly,

$$
(\hat{E}(\varDelta)f, g) = (f, E(\varDelta)g) \quad (f \in \mathfrak{H}_-,\, g \in \mathfrak{H}), \tag{22}
$$

One can easily see that $\hat{E}(\varDelta)f = E(\varDelta)f$, $\forall f \in \mathfrak{H}$, so that $\hat{E}(\varDelta)$ is the extension of $E(\varDelta)$ by continuity. We say that $\hat{E}(\varDelta)$, as a function of $\varDelta \in \aleph$, is the *extended generalized (canonical) spectral function* of A corresponding to the self-adjoint extension $\tilde{A}$ (or to the original spectral function $E(\varDelta)$). It is known [19] that $\hat{E}(\varDelta) \in [\mathfrak{H}_-, \mathfrak{H}_+]$, $\forall \varDelta \in \aleph$, and $(\hat{E}(\varDelta)f, f) \geq 0$ for all $f \in \mathfrak{H}_-$. It is also known [19] that the complex scalar measure $(E(\varDelta)f, g)$ is a complex function of bounded variation on the real axis. However, $(\hat{E}(\varDelta)f, g)$ may be unbounded for $f, g \in \mathfrak{H}_-$.

Now let $\hat{R}_\lambda$ be an extended generalized (canonical) resolvent of a closed Hermitian operator A and let $\hat{E}(\varDelta)$ be the corresponding extended generalized (canonical) spectral function. It was shown in [19] that for any $f, g \in \mathfrak{H}_-$,

$$
\int\limits_{-\infty}^{+\infty} \frac{|d(\hat{E}(\varDelta)f, g)|}{1+t^2} < \infty, \tag{23}
$$

and the following integral representation holds

$$
\hat{R}_\lambda - \frac{\hat{R}_i + \hat{R}_{-i}}{2} = \int\limits_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\hat{E}(t). \tag{24}
$$

Lemma 6. *Let* $\mathbb{A} = A^* + \mathcal{R}^{-1}(S - \frac{i}{2}P^+_{\mathfrak{N}_i} + \frac{i}{2}P^+_{\mathfrak{N}_{-i}})P^+_{\mathfrak{M}}$ *be a strong self-adjoint bi-extension of a regular Hermitian operator* A *with the quasi-kernel* $\hat{A}$ *and let* $\hat{E}(\Delta)$ *be the extended canonical spectral function of* $\hat{A}$*. Then for every* $f \in \mathfrak{H} \oplus L$, $f \neq 0$, *and for every* $g \in \mathfrak{H}_-$ *there is an integral representation*

$$
(\bar{R}_\lambda f, g) = \int\limits_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d(\hat{E}(t)f, g) + \frac{1}{2}((\hat{R}_i + \hat{R}_{-i})f, g). \tag{25}
$$

Here $F = \mathfrak{H}_+ \ominus \mathfrak{D}(A)$, $L = \mathcal{R}^{-1}(S - \frac{i}{2}P^+_{\mathfrak{N}_i} + \frac{i}{2}P^+_{\mathfrak{N}_{-i}})F$, $\bar{R}_\lambda = \overline{(\mathbb{A} - \lambda I)^{-1}}$.

Theorem 7. *Let* $\mathbb{A} = A^* + \mathcal{R}^{-1}(S - \frac{i}{2}P^+_{\mathfrak{N}_i} + \frac{i}{2}P^+_{\mathfrak{N}_{-i}})P^+_{\mathfrak{M}}$ *be a strong self-adjoint bi-extension of a regular Hermitian operator* A *with the quasi-kernel* $\hat{A}$ *and let* $\hat{E}(\Delta)$ *be the extended canonical spectral function of* $\hat{A}$*. Also, let* $F = \mathfrak{H}_+ \ominus \mathfrak{D}(A)$ *and* $L = \mathcal{R}^{-1}(S - \frac{i}{2}P^+_{\mathfrak{N}_i} + \frac{i}{2}P^+_{\mathfrak{N}_{-i}})F$*. Then for every* $f \in L \dotplus \mathfrak{L}$ *with* $f \neq 0$ *and* $f \in \mathfrak{R}(\mathbb{A} - \lambda I)$*, we have*

$$
\int\limits_{-\infty}^{+\infty} d(\hat{E}(t)f, f) = \infty, \qquad \text{if} \quad f \notin \mathfrak{L}, \tag{26}
$$

and

$$
\int\limits_{-\infty}^{+\infty} d(\hat{E}(t)f, f) < \infty, \qquad \text{if} \quad f \in \mathfrak{L}. \tag{26'}
$$

Moreover, there exist real constants b *and* c *such that*

$$
c\|f\|^2_- \leq \int\limits_{-\infty}^{+\infty} \frac{d(\hat{E}(t)f, f)}{1+t^2} \leq b\|f\|^2_-, \tag{27}
$$

for all $f \in L \dotplus \mathfrak{L}$.

Corollary 3. *In the settings of Theorem 7 for all* $f, g \in L \dotplus \mathfrak{L}$

$$
\left| \left(\frac{\hat{R}_i + \hat{R}_{-i}}{2} f, g \right) \right| \leq a \sqrt{\int_{-\infty}^{+\infty} \frac{d(\hat{E}(t)f, f)}{1+t^2}} \cdot \sqrt{\int_{-\infty}^{+\infty} \frac{d(\hat{E}(t)g, g)}{1+t^2}}, \tag{28}
$$

where $a > 0$ *is a constant (see* [2]*).*

3. Linear Stationary Conservative Dynamic Systems

In this section we consider linear stationary conservative dynamic systems (l. s. c. d. s.) θ of the form

$$(29)\qquad \begin{cases} (\mathbb{A} - zI) = KJ\varphi_- \\ \varphi_+ = \varphi_- - 2iK^*x \end{cases} \qquad (\mathrm{Im}\ \mathbb{A} = KJK^*)\,.$$

In a system θ of the form (29) $\mathbb{A}$, K and J are bounded linear operators in Hilbert spaces, φ_- is an input vector, φ_+ is an output vector, and x is an inner state vector of the system θ. For our purposes we need the following more precise definition:

Definition. *The array*

$$(30)\qquad \theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & E \end{pmatrix}$$

is called a linear stationary conservative dynamic system or Brodskiĭ-Livšic rigged operator colligation if

(1) $\mathbb{A}$ *is a correct* $(*)$*-extension of an operator* T *of the class* Λ_A.
(2) $J = J^* = J^{-1} \in [E, E]$, $\quad dim\, E < \infty$
(3) $\mathbb{A} - \mathbb{A}^* = 2iKJK^*$, *where* $K \in [E, \mathfrak{H}_-]$ $\qquad (K^* \in [\mathfrak{H}_+, E])$

In this case, the operator K is called a *channel operator* and J is called a *direction operator*. A system θ of the form (30) will be called a *scattering* system (*dissipative* operator colligation) if $J = I$. We will associate with the system θ the operator-valued function

$$(31)\qquad W_\theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ$$

which is called the *transfer operator-valued function* of the system θ or the characteristic operator-valued function of Brodskiĭ-Livšic rigged operator colligation. According to Theorem 7, $\mathfrak{R}(K) \subset \mathfrak{R}(\mathbb{A} - \lambda I)$ and therefore $W_\theta(z)$ is well-defined. It may be shown [10], [25] that the transfer operator-function of the system θ of the form (30) has the following properties:

$$(32)\qquad \begin{aligned} W_\theta^*(z)JW_\theta(z) - J &\geq 0 \quad (\mathrm{Im}\ z > 0, z \in \rho(T)), \\ W_\theta^*(z)JW_\theta(z) - J &= 0 \quad (\mathrm{Im}\ z = 0, z \in \rho(T)), \\ W_\theta^*(z)JW_\theta(z) - J &\leq 0 \quad (\mathrm{Im}\ z < 0, z \in \rho(T)), \end{aligned}$$

where $\rho(T)$ is the set of regular points of an operator T. Similar relations take place if we change $W_\theta(z)$ to $W_\theta^*(z)$ in (32). Thus, the transfer operator-valued function of the system

θ of the form (30) is J-contractive in the lower half-plane on the set of regular points of an operator T and J-unitary on real regular points of an operator T.

Let θ be a l.s.c.d.s. of the form (30). We consider the operator-valued function

$$V_\theta(z) = K^*(\mathbb{A}_R - zI)^{-1}K. \tag{33}$$

The transfer operator-function $W_\theta(z)$ of the system θ and an operator-function $V_\theta(z)$ of the form (33) are connected with the relation

$$V_\theta(z) = i[W_\theta(z) + I]^{-1}[W_\theta(z) - I]J. \tag{34}$$

As it is known [11] an operator-function $V(z) \in [E, E]$ is called an *operator-valued R-function* if it is holomorphic in the upper half-plane and $\mathrm{Im}\, V(z) \geq 0$ whenever $\mathrm{Im}\, z > 0$.

It is known [11,17] that an operator-valued R-function acting on a Hilbert space E ($\dim E < \infty$) has an integral representation

$$V(z) = Q + F \cdot z + \int_{-\infty}^{+\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) dG(t), \tag{35}$$

where $Q = Q^*$, $F \geq 0$ in the Hilbert space E, and $G(t)$ is a non-decreasing operator-function on $(-\infty, +\infty)$ for which

$$\int_{-\infty}^{+\infty} \frac{dG(t)}{1 + t^2} \in [E, E].$$

Definition. *We call an operator-valued R-function $V(z)$ acting on a Hilbert space E, ($\dim E < \infty$) realizable if in some neighborhood of the point $(-i)$, the function $V(z)$ can be represented in the form*

$$V(z) = i[W_\theta(z) + I]^{-1}[W_\theta(z) - I]J, \tag{36}$$

where $W_\theta(z)$ is the transfer operator-function of some l.s.c.d.s. θ with the direction operator J ($J = J^* = J^{-1} \in [E, E]$).

Definition. *An operator-valued R-function $V(z) \in [E, E]$, ($\dim E < \infty$) is said to be a member of the class $N(R)$ if in the representation (35) we have*

$$i) \qquad F = 0,$$

$$ii) \qquad Qe = \int_{-\infty}^{+\infty} \frac{t}{1 + t^2} dG(t)e,$$

for all $e \in E$ with

$$\int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty.$$

We now establish the next result.

Theorem 8. *Let θ be a l.s.c.d.s. of the form* (30) *with* $\dim E < \infty$*. Then the operator-function $V_\theta(z)$ of the form* (33), (34) *belongs to the class $N(R)$.*

Proof. Let G_{-i} be a neighborhood of $(-i)$ and $\lambda, \mu \in G_{-i}$. Then,

$$\begin{aligned} V_\theta(\lambda) - V_\theta(\mu) &= K^*(\mathbb{A}_R - \lambda I)^{-1}K - K^*(\mathbb{A}_R - \mu I)^{-1}K \\ &= (\mu - \lambda)K^*(\mathbb{A}_R - \lambda I)^{-1}(\mathbb{A}_R - \mu I)^{-1}K, \end{aligned} \tag{37}$$

and

$$\frac{V_\theta(\lambda) - V_\theta(\mu)}{\mu - \lambda} = K^*(\mathbb{A}_R - \lambda I)^{-1}(\mathbb{A}_R - \mu I)^{-1}K, \tag{38}$$

for all $\lambda, \mu \in G_{-i}$. Therefore, letting $\lambda \to \mu$ we can say that $V_\theta(z)$ is holomorphic in G_{-i}. Without loss of generality (see [25]) we can conclude that $V_\theta(z)$ is holomorphic in any one of the half-planes.

It is obvious that $V_\theta^*(z) = \overline{V_\theta(z)} = V_\theta(\bar z)$. Furthermore,

$$\mathrm{Im} V_\theta(z) = \frac{1}{2i}K^*(\mathbb{A}_R - \bar z I)^{-1}(\mathbb{A}_R - zI)^{-1}K. \tag{39}$$

Since $(-i)$ is a regular point of the operator T in the system (30) then (see [10]) $I + iV(\lambda)J$ is invertible in G_{-i}.

Let now $D_z = (\mathbb{A}_R - zI)^{-1}K$, then it is easy to see that the adjoint operator D_z^* is given by $D_z^* = K^*(\mathbb{A}_R - \bar z I)^{-1}$. Therefore, we have $\mathrm{Im} V_\theta(z) = \mathrm{Im} z D_z^* D_z$ which implies that $\mathrm{Im} V_\theta(z) \geq 0$ when $\mathrm{Im} z > 0$. Hence we can conclude that $V_\theta(z)$ is an operator R-function and admits representation (35).

Let now $B = K^*(\mathbb{A}_R + iI)^{-1}(\mathbb{A}_R - iI)^{-1}K$. It follows from (39) that $B = \frac{1}{2i}(V_\theta(i) - V_\theta^*(i))$. Using Theorem 7 and representation (35) one can show that

$$Bf = \int_{-\infty}^{\infty} \frac{dG(t)}{1+t^2} f, \qquad f \in E \tag{40}$$

and $B \in [E, E]$.

Let $\hat E(\varDelta)$ be the canonical extended spectral function of the quasi-kernel $\hat A$ of the operator $\mathbb{A}_R = \frac{1}{2}(\mathbb{A} + \mathbb{A}^*)$. Then relying on Lemma 6 for all $f, g \in E$ we have

$$(V_\theta(\lambda)f, g)_E = \int_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right) d(\hat G(t)f, g)_E + (\hat Q f, g)_E, \tag{41}$$

where $\hat{G}(\mathit{\Delta}) = K^* \hat{E}(\mathit{\Delta}) K$, $\mathit{\Delta} \in \aleph$ and

$$\hat{Q} = \frac{1}{2} K^* [(\mathbb{A}_R - iI)^{-1} + (\mathbb{A}_R + iI)^{-1}] K = \frac{1}{2} [V_\theta(-i) + V_\theta^*(-i)]. \tag{42}$$

From Theorem 7 (see also [19]), we have for all $f \in E$ with $Kf \in \mathfrak{L}$,

$$\int\limits_{-\infty}^{\infty} d(\hat{G}(t)f, f)_E < \infty, \tag{43}$$

and

$$c \|Kf\|_-^2 \leq \int\limits_{-\infty}^{+\infty} \frac{d(\hat{G}(t)f, f)_E}{1+t^2} \leq b \|Kf\|_-^2. \tag{44}$$

Moreover, (28) implies that

$$\left| \left(\hat{Q}f, g \right)_E \right| \leq C \sqrt{\int_{-\infty}^{+\infty} \frac{d(\hat{G}(t)f, f)_E}{1+t^2}} \cdot \sqrt{\int_{-\infty}^{+\infty} \frac{d(\hat{G}(t)g, g)_E}{1+t^2}}. \tag{45}$$

By (41) we have for any $f, g \in E$

$$(V_\theta(\lambda)f, g)_E = (\hat{Q}f, g)_E + \int\limits_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d(\hat{G}(t)f, g)_E. \tag{46}$$

On the other hand (35) implies

$$(V_\theta(\lambda)f, g)_E = (Qf, g)_E + \lambda (Ff, g)_E + \int\limits_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d(G(t)f, g)_E. \tag{47}$$

Comparing (46) and (47) we get $(Qf, g)_E = (\hat{Q}f, g)_E$, $(Ff, g)_E = 0$, and $(G(\Delta)f, g) = (\hat{G}(\Delta)f, g)$ $(\Delta \in \aleph)$, for all $f, g \in E$. Taking into account the continuity and positivity of F, $G(\Delta)$, and $\hat{G}(\Delta)$, we find that $F = 0$ and $G(\Delta) = \hat{G}(\Delta)$ $(\Delta \in \aleph)$.

Thus,

$$V(\lambda) = Q + \int\limits_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) dG(t), \tag{48}$$

holds.

Let $E_\infty = K^{-1}\mathfrak{L}$, $E_\infty \subset E$. Since $\hat{E}(\Delta)$ coincides with $E(\Delta)$ on $\mathfrak{L}$, then for any $e \in E_\infty$, we have

$$\int_{-\infty}^{+\infty} d(\hat{G}(t)e, e)_E < \infty. \tag{49}$$

If $e \notin E_\infty$, then $Ke \notin \mathfrak{L}$ (see Theorem 7) and

$$\int_{-\infty}^{+\infty} d(\hat{G}(t)e, e)_E = \infty. \tag{50}$$

Further, since

$$Q = \frac{1}{2}\left[V_\theta(i) + V_\theta(-i)\right] = \frac{1}{2}\left[K^*((\mathbb{A}_R + iI)^{-1} + (\mathbb{A}_R - iI)^{-1})K\right], \tag{51}$$

we have $\mathfrak{R}(Q) \subseteq \mathfrak{R}(K^*) \subseteq E$. Now formula (45) yields

$$|(Qf, g)_E| \leq C\|f\|_E \cdot \|g\|_E, \qquad f, g \in E. \tag{52}$$

On the other hand, if $e \in E_\infty$ then

$$\begin{aligned} Qe &= \frac{1}{2}\left[K^*(\hat{A}_R + iI)^{-1} + (\hat{A}_R - iI)^{-1})Ke\right] \\ &= K^* \int_{-\infty}^{+\infty} \frac{t}{1+t^2}\, dE(t)Ke = \int_{-\infty}^{+\infty} \frac{t}{t^2+1}\, d\hat{G}(t)e. \end{aligned}$$

This completes the proof.

Next, we establish the converse.[2]

Theorem 9. *Let an operator-valued function $V(z)$ act on a finite-dimensional Hilbert space E and belong to the class $N(R)$. Then $V(z)$ admits a realization by the system θ of the form (30) with a preassigned direction operator J for which $I + iV(-i)J$ is invertible.*

Proof. We will use several steps to prove this theorem.
Step 1. Let $C_{00}(E, (-\infty, +\infty))$ be the set of continuous compactly supported vector-valued functions $f(t)$ $(-\infty < t < +\infty)$ with values in a finite dimensional Hilbert space E. We introduce an inner product $(\cdot, \cdot)$ defined by

$$(f, g) = \int_{-\infty}^{+\infty} (G(dt)f(t), g(t))_E \tag{53}$$

[2]The method of rigged Hilbert spaces for solving inverse problems in the theory of characteristic operator-valued functions was introduced in [23] and was developed further in [2].

for all $f, g \in C_{00}(E, (-\infty, +\infty))$. In order to construct a Hilbert space, we identify with zero all functions $f(t)$ such that $(f, f) = 0$. Then we make the completion and obtain the new Hilbert space $L^2_G(E)$. Let us note that the set $C_{00}(E, (-\infty, +\infty))$ is dense in $L^2_G(E)$. Moreover, if $f(t)$ is continuous and

$$\int_{-\infty}^{+\infty} (G(dt)f(t), f(t))_E < \infty, \tag{54}$$

then $f(t)$ belongs to $L^2_G(E)$.

Let $\mathfrak{D}_0$ be the set of the continuous vector-valued (with values in E) functions $f(t)$ such that in addition to (54), we have

$$\int_{-\infty}^{+\infty} t^2 (G(dt)f(t), f(t))_E < \infty. \tag{55}$$

Since $C_{00} \subset \mathfrak{D}_0$, it follows that $\mathfrak{D}_0$ is dense in $L^2_G(E)$. We introduce an operator $\hat{A}$ on $\mathfrak{D}_0$ in the following way:

$$\hat{A}f(t) = tf(t). \tag{56}$$

Below we denote again by $\hat{A}$ the closure of the Hermitian operator $\hat{A}$ (56). It is easy to see that this operator is Hermitian. Now $\hat{A}$ is a self-adjoint operator in $L^2_G(E)$ (see [9]).

Let $\tilde{\mathfrak{H}}_+ = \mathfrak{D}(\hat{A})$ and define the inner product

$$(f, g)_{\tilde{\mathfrak{H}}_+} = (f, g) + (\hat{A}f, \hat{A}g) \tag{57}$$

for all $f, g \in \tilde{\mathfrak{H}}_+$. It is clear that $\tilde{\mathfrak{H}}_+$ is a Hilbert space with norm $\|\cdot\|_{\tilde{\mathfrak{H}}_+}$ generated by the inner product (57). We equip the space $L^2_G(E)$ with spaces $\tilde{\mathfrak{H}}_+$ and $\tilde{\mathfrak{H}}_-$:

$$\tilde{\mathfrak{H}}_+ \subset L^2_G(E) \subset \tilde{\mathfrak{H}}_-. \tag{58}$$

Let us denote by $\tilde{\mathcal{R}}$ the corresponding Riesz-Berezanskii operator, $\tilde{\mathcal{R}} \in [\tilde{\mathfrak{H}}_-, \tilde{\mathfrak{H}}_+]$.

Consider the following subspaces of the space E:

$$\begin{aligned} E_\infty &= \{e \in E : \int_{-\infty}^{+\infty} d(G(t)e, e)_E < \infty\} \\ F_\infty &= E_\infty^\perp. \end{aligned} \tag{59}$$

If $e \in E_\infty$, then (54) implies that the function $e(t) = e$ is an element of the space $L^2_G(E)$. On the other hand, if $e \in E$ and $e \notin E_\infty$ then $e(t)$ does not belong to $L^2_G(E)$. It can be shown that any function $e(t) = e \in E$ can be identified with an element of $\tilde{\mathfrak{H}}_-$. Indeed, since for all $e \in E$

(60) $$\int_{-\infty}^{+\infty} \frac{d(G(t)e, e)_E}{1+t^2} < \infty,$$

the function

(61) $$\tilde{e}(t) = \frac{e}{\sqrt{1+t^2}}$$

belongs to the space $L_G^2(E)$. Letting $f(t) \in \mathfrak{D}_0$, we have

(62) $$\int_{-\infty}^{+\infty} (1+t^2)(G(dt)f(t), f(t))_E < \infty.$$

Therefore, the function $\tilde{f}(t) = \sqrt{1+t^2}f(t)$ belongs to the space $L_G^2(E)$ and hence

$$(\tilde{f}(t), \tilde{e}(t)) = \int_{-\infty}^{+\infty} (G(dt)\tilde{f}(t), \tilde{e}(t))_E.$$

Furthermore,

(63) $$\begin{aligned} |(\tilde{f}(t), \tilde{e}(t))| &\leq \|\tilde{f}(t)\| \cdot \|\tilde{e}(t)\| \\ &= \sqrt{\int_{-\infty}^{+\infty} (1+t^2)(G(dt)f(t), f(t))_E} \cdot \sqrt{\int_{-\infty}^{+\infty} \frac{d(G(t)\tilde{e}(t), \tilde{e}(t))}{1+t^2}}e \\ &= \|f\|_{\tilde{\mathfrak{H}}_+} \cdot \|e\|_E. \end{aligned}$$

Also,

$$\begin{aligned} \int_{-\infty}^{+\infty} (G(dt)f(t), e(t))_E &= \int_{-\infty}^{+\infty} \left(\sqrt{1+t^2}G(dt)f(t), \frac{e}{\sqrt{1+t^2}}\right)_E \\ &= \int_{-\infty}^{+\infty} (G(dt)\tilde{f}(t), \tilde{e}(t))_E \\ &= (\tilde{f}(t), \tilde{e}(t)). \end{aligned}$$

Therefore,

(64) $$e(f) = \int_{-\infty}^{+\infty} (G(dt)f(t), e(t))_E$$

is a continuous linear functional over $\tilde{\mathfrak{H}}_+$, for $f \in \mathfrak{D}_0$. Since $\mathfrak{D}_0$ is dense in $\tilde{\mathfrak{H}}_+$, $e(t) = e$ belongs to $\tilde{\mathfrak{H}}_-$.

We calculate the Riesz-Berezanskii mapping on the vectors $e(t) = e$, $e \in E$. By the definition of $\tilde{\mathcal{R}}$, for all $f \in \tilde{\mathfrak{H}}_+$ we have $(f, e) = (f, \tilde{\mathcal{R}}e)_{\tilde{\mathfrak{H}}_+}$. Hence, for all $f \in \mathfrak{D}_0$ (see also [2])

$$(f, e) = \int_{-\infty}^{+\infty} (G(dt)f(t), e(t))_E = \int_{-\infty}^{+\infty} (1+t^2) \left(G(dt)f(t), \frac{e(t)}{1+t^2} \right)_E$$
$$= \left(f, \frac{e(t)}{1+t^2} \right)_{\tilde{\mathfrak{H}}_+} = (f, \tilde{\mathcal{R}}e)_{\tilde{\mathfrak{H}}_+}.$$

Thus

$$\tilde{\mathcal{R}}e = \frac{e(t)}{1+t^2}, \quad e \in E. \tag{65}$$

Let us note some properties of the operator $\hat{A}$. It is easy to see that for all $g \in \tilde{\mathfrak{H}}_+$, we have that $\|\hat{A}g\| \leq \|g\|_{\tilde{\mathfrak{H}}_+}$. Taking this into account we obtain

$$\|\hat{A}f\|_{\tilde{\mathfrak{H}}_-} = \sup_{g \in \tilde{\mathfrak{H}}_+} \frac{|(\hat{A}f, g)|}{\|g\|_{\tilde{\mathfrak{H}}_+}} = \sup_{g \in \tilde{\mathfrak{H}}_+} \frac{|(f, \hat{A}g)|}{\|g\|_{\tilde{\mathfrak{H}}_+}} \leq \sup_{g \in \tilde{\mathfrak{H}}_+} \frac{\|f\| \cdot \|\hat{A}g\|}{\|g\|_{\tilde{\mathfrak{H}}_+}} \leq \|f\|. \tag{66}$$

Hence, the operator $\hat{A}$ is $(\cdot, -)$-continuous. Let $\overline{\hat{A}}$ be the extension of the operator $\hat{A}$ to $\mathfrak{H}$ with respect to $(\cdot, -)$-continuity. Now,

$$(\overline{\hat{A}} - \lambda I)^{-1} g - (\overline{\hat{A}} - \mu I)^{-1} g = (\lambda - \mu)(\overline{\hat{A}} - \lambda I)^{-1} (\overline{\hat{A}} - \mu I)^{-1} g \tag{67}$$

holds for all $g \in \tilde{\mathfrak{H}}_-$. Note in particular that

$$(\overline{\hat{A}} - iI)^{-1} g - (\overline{\hat{A}} + iI)^{-1} g = 2i(\overline{\hat{A}} - iI)^{-1} (\overline{\hat{A}} + iI)^{-1} g \tag{68}$$

and

$$\|(\overline{\hat{A}} - iI)^{-1} g\|^2 = \|(\overline{\hat{A}} + iI)^{-1} g\|^2 \tag{69}$$

for all g in $\tilde{\mathfrak{H}}_-$. It follows from (60) that the element

$$f(t) = \frac{f}{t - \lambda}, \quad f \in E \tag{70}$$

belongs to the space $L^2_G(E)$. It is easy to show that, for all $e \in E$,

$$(\overline{\hat{A}} - \lambda I)^{-1} e = \frac{e}{t - \lambda}, \qquad (\mathrm{Im}\lambda \neq 0). \tag{71}$$

STEP 2. Now let $\tilde{\mathfrak{H}}_+$ be the Hilbert space constructed in Step 1 and let

$$\mathfrak{D}(A) = \tilde{\mathfrak{H}}_+ \ominus \tilde{\mathcal{R}}E, \tag{72}$$

where by $\ominus$ we mean orthogonality in $\tilde{\mathfrak{H}}_+$. We define an operator A on $\mathfrak{D}(A)$ by the following expression:

$$A = \hat{A}\Big|_{\mathfrak{D}(A)}. \tag{73}$$

Obviously, A is a closed Hermitian operator.

Let us note that if $E_\infty = 0$ then $\mathfrak{D}(A)$ is dense in $L^2_G(E)$. Define $\mathfrak{H}_0 = \overline{\mathfrak{D}(A)}$ and let P be the orthogonal projection of $\mathfrak{H} = L^2_G(E)$ onto $\mathfrak{H}$. We shall show that PA and $P\hat{A}$ are closed operators in $\mathfrak{H}$. Let

$$A_1 = \hat{A}\Big|_{\mathfrak{D}(A_1)}, \qquad \mathfrak{D}(A_1) = \tilde{\mathfrak{H}}_+ \ominus \tilde{\mathcal{R}}E_\infty. \tag{74}$$

The following obvious inclusions hold: $A \subset A_1 \subset \hat{A}$. It is easy to see that $\mathfrak{D}(A_1) = \mathfrak{D}(A) \oplus \tilde{\mathcal{R}}F_\infty$, $\overline{\mathfrak{D}(A_1)} = \mathfrak{H}_0$ and A_1 is a closed Hermitian operator. Indeed, if we identify the space E with the space of functions $e(t) = e$, $e \in E$ we would obtain $L^2_G(E) \ominus \mathfrak{H}_0 = E_\infty$. Since

$$\int_{-\infty}^{+\infty} \frac{d(G(t)e, h)_E}{1+t^2} = 0$$

and

$$\tilde{\mathcal{R}}\tilde{e} = \frac{\tilde{e}}{1+t^2}, \qquad \tilde{e} \in F_\infty$$

for all $e \in E_\infty$, $h \in F_\infty$, we find that E_∞ is $(\cdot)$-orthogonal to $\mathcal{R}F_\infty$ and hence $\overline{\mathfrak{D}(A_1)} = \mathfrak{H}_0$.

We denote by A_1^* the adjoint of the operator A_1. Now we are going to find the defect subspaces $\mathfrak{N}_i$ and $\mathfrak{N}_{-i}$ of the operator A. Since the subspace $E \in \tilde{\mathfrak{H}}_-$ is $(\cdot)$-orthogonal to $\mathfrak{D}(A)$, we have that $(\overline{\hat{A}} \pm iI)^{-1}E = \mathfrak{N}_{\pm i}$. Moreover, by (71) we have

$$(\overline{\hat{A}} \pm iI)^{-1}e = \frac{e}{t \pm i}, \qquad e \in E. \tag{75}$$

Therefore

$$\mathfrak{N}_{\pm i} = \left\{ f(t) \in L^2_G(E),\ f(t) = \frac{e}{t \pm i}, \quad e \in E \right\}. \tag{76}$$

Similarly, the defect subspaces of the operator A_1 are

$$\mathfrak{N}^0_{\pm i} = \left\{ f(t) \in L^2_G(E),\ f(t) = \frac{e}{t \pm i},\ e \in E_\infty \right\}. \tag{77}$$

Obviously, $\mathfrak{N}_\lambda^0 \subset \mathfrak{D}_0$ because

$$\int\limits_{-\infty}^{+\infty} \frac{t}{|t-\lambda|^2}(G(dt)e,e)_E \leq K(\lambda)\int\limits_{-\infty}^{+\infty}(G(dt)e,e)_E < \infty, \quad e \in E_\infty.$$

Taking into account that

$$\mathfrak{D}(A_1^*) = \mathfrak{D}(A) \dotplus \mathfrak{N}_i^0 \dotplus \mathfrak{N}_{-i}^0, \tag{78}$$

we can conclude that $\mathfrak{D}(A_1^*) \subseteq \mathfrak{D}(\hat{A})$. At the same time, the inclusion $A_1 \subset \hat{A}$ implies that $\mathfrak{D}(A_1^*) \supset \mathfrak{D}(\hat{A})$. Combining these two we obtain $\mathfrak{D}(A_1^*) = \mathfrak{D}(\hat{A})$ and $P\hat{A} = A_1^*$. Since A_1^* is a closed operator, $P\hat{A}$ is also closed. Consequently, $\hat{A}$ is the regular self-adjoint extension of the operator A which implies A is a regular Hermitian operator.

Since $\hat{A}$ is the self-adjoint extension of operator A we find by (10) that

$$\mathfrak{D}(\hat{A}) = \mathfrak{D}(A) \dotplus (I-U)\mathfrak{N}_i \tag{79}$$

for some admissible isometric operator U acting from $\mathfrak{N}_i$ into $\mathfrak{N}_{-i}$. It is easy to check that $U(\overline{\hat{A}}-iI)^{-1}e = (\overline{\hat{A}}+iI)^{-1}e$, for all e in E. Consequently, the operator U has the form:

$$U\left(\frac{e}{t-i}\right) = \frac{e}{t+i}, \quad e \in E. \tag{80}$$

Straightforward calculations show that

$$\hat{A}(I-U)\left(\frac{e}{t-i}\right) = t\frac{e}{t-i} - t\frac{e}{t+i} = \frac{2ite}{t^2+1}.$$

Let A^* be the adjoint of the operator A. In the space $\mathfrak{D}(A^*) = \mathfrak{H}_+$ we introduce an inner product

$$(f,g)_+ = (f,g) + (A^*f, A^*g), \tag{81}$$

and construct the rigged space $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$ with corresponding Riesz-Berezanskii operator $\mathcal{R}$. Since $P\hat{A}$ is a closed Hermitian operator, $\tilde{\mathfrak{H}}_+$ is a subspace of $\mathfrak{H}_+$.

By Theorem 2, $\mathfrak{H}_+ = \mathfrak{D}(\hat{A}) \dotplus (U-I)\hat{\mathfrak{N}}_i$, where

$$\hat{\mathfrak{N}}_i = \{f_i \in \mathfrak{N}_i, \ (U-I)f_i \in \mathfrak{H}_0\}.$$

Taking into account that

$$(U-I)\left(\frac{e}{t-i}\right) = \frac{-2ie}{t^2+1}, \quad e \in E,$$

we can conclude that

$$\hat{\mathfrak{N}}_i = \left\{ \frac{\tilde{e}}{t-i}, \;\; e \in F_\infty = E \ominus E_\infty \right\}.$$

Therefore,

$$\mathfrak{D}(A^*) = \mathfrak{D}(\hat{A}) \dotplus \left\{ \frac{t\tilde{e}}{t^2+1} \right\}, \quad e \in F_\infty. \tag{82}$$

Step 3. In this Step we will construct a special self-adjoint bi-extension whose quasi-kernel coincides with the operator $\hat{A}$. Then applying (7), we will have

$$\mathfrak{H}_+ = \mathfrak{D}(A) \oplus \mathfrak{N}'_i \oplus \mathfrak{N}'_{-i} \oplus \mathfrak{N},$$

where $\mathfrak{N}'_{\pm i}$ are semidefect spaces of the operator A, $\mathfrak{N} = \mathcal{R}E_\infty$, and

$$\mathfrak{D}(A) \oplus E_\infty = \mathfrak{H} = L^2_G(E).$$

We begin by setting

$$(f,g)_1 = (f,g)_+ + (P^+_{\mathfrak{N}} f, P^+_{\mathfrak{N}} g)_+, \quad \text{for all} \;\; f,g \in \mathfrak{H}_+. \tag{83}$$

Here $P^+_{\mathfrak{N}}$ is an orthoprojection of $\mathfrak{H}_+$ onto $\mathfrak{N}$. Obviously, the norm $\|\cdot\|_1$ is equivalent to $\|\cdot\|_+$. We denote by $\mathfrak{H}_{+1}$ the space $\mathfrak{H}_+$ with the norm $\|\cdot\|_1$, so that $\mathfrak{H}_{+1} \subset \mathfrak{H} \subset \mathfrak{H}_{-1}$ is the corresponding rigged space with Riesz-Berezanskii operator $\mathcal{R}_1$.

By Theorem 1 there exists a (1)-isometric operator V such that

$$\mathfrak{D}(\hat{A}) = \mathfrak{D}(A) \oplus (V+I)(\mathfrak{N}'_i \oplus \mathfrak{N}), \tag{84}$$

where $\mathfrak{D}(V) = \mathfrak{N}'_i \oplus \mathfrak{N}$, $\mathfrak{R}(V) = \mathfrak{N}'_{-i} \oplus \mathfrak{N}$ and (-1) is a regular point for the operator V. Moreover,

$$\begin{cases} \varphi = i(I + P^+_{\mathfrak{N}'_i})(A^* + iI)^{-1} f_i, \\ V\varphi = i(I + P^+_{\mathfrak{N}'_{-i}})(A^* - iI)^{-1} U f_i, \\ \text{where } \varphi \in \mathfrak{D}(V), \; f_i \in \mathfrak{N}_i. \end{cases} \tag{85}$$

Here U is the isometric operator described in Step 2. Consequently we obtain

$$\begin{cases} f_i = \frac{i}{2}(A^* + iI)(I + P^+_{\mathfrak{N}})\varphi, \\ U f_i = -\frac{i}{2}(A^* - iI)(I + P^+_{\mathfrak{N}}) V\varphi, \\ \text{where } \varphi \in \mathfrak{D}(V), \; f_i \in \mathfrak{N}_i. \end{cases} \tag{86}$$

It follows that

$$
\begin{aligned}
f_i - Uf_i &= \varphi + V\varphi + iA^* P^+_{\mathfrak{N}}(V-I)\varphi \\
\hat{A}(f_i - Uf_i) = i(I+U)f_i &= A^*(\varphi + V\varphi) + iP^+_{\mathfrak{N}}(I-V)\varphi \\
f_i + Uf_i &= \varphi - V\varphi - iA^* P^+_{\mathfrak{N}}(I-V)\varphi
\end{aligned}
$$

Applying formula (11) we get

$$\mathfrak{H}_+ = \mathfrak{D}(\hat{A}) \dotplus (U+I)\tilde{\mathfrak{N}}_i, \text{ and } \tilde{\mathfrak{N}}_i = \{f_i \in \tilde{\mathfrak{N}}_i \ (U-I)f_i \in \mathfrak{H}\}.$$

Since $f_i - Uf_i = \varphi + V\varphi + iA^* P^+_{\mathfrak{N}}(V-I)\varphi$, we find that $f_i - Uf_i \in \mathfrak{H}$ if and only if $P^+_{\mathfrak{N}}(V+I)\varphi = 0$. (This follows from the fact that $A^* P^+_{\mathfrak{N}}(V-I)\varphi \in \mathfrak{D}(A) \subset \mathfrak{H}$ and from the formula $\mathfrak{H} = \mathfrak{H}_0 \dotplus \mathfrak{N}$ (see [4])). Let us note that if $P^+_{\mathfrak{N}}(V+I)\varphi = 0$ then $f_i + Uf_i = \varphi - V\varphi$. Thus,

$$\tilde{\mathfrak{N}}_i = \{f = (A^* + iI)(I + P^+_{\mathfrak{N}})\varphi, \ \ P^+_{\mathfrak{N}}(V+I)\varphi = 0\}. \tag{87}$$

Let $N = \mathrm{Ker} P^+_{\mathfrak{N}}(I+V)$. Then we have

$$\mathfrak{H}_+ = \mathfrak{D}(\hat{A}) \dotplus (I-V)N. \tag{88}$$

We denote by P_0 the projection operator of $\mathfrak{H}_+$ onto $\mathfrak{D}(\hat{A})$ along $(I-V)N$, $P_1 = I - P_0$. Since $\mathfrak{D}(\hat{A}) = \tilde{\mathfrak{H}}_+$, we have $P_0 \in [\mathfrak{H}_+, \tilde{\mathfrak{H}}_+]$. We will denote by $P_0^* \in [\tilde{\mathfrak{H}}_-, \mathfrak{H}_-]$ the adjoint operator to P_0, i.e. $(P_0 f, g) = (f, P_0^* g)$, for all $f \in \mathfrak{H}_+$, $g \in \mathfrak{H}_-$. If $\tilde{f}_i \in \tilde{\mathfrak{N}}_i$, then $\tilde{f}_i + U\tilde{f}_i = (I-V)\varphi$, for $\varphi \in N$, and

$$
\begin{aligned}
A^*(I-V)\varphi &= iP^+_{\mathfrak{N}'_i}\varphi + iP^+_{\mathfrak{N}'_{-i}}V\varphi + AP^+_{\mathfrak{N}}(I-V)\varphi = i(V+I)\varphi + A^* P^+_{\mathfrak{N}}(I-V)\varphi \\
&= i[(I+V)\varphi - iA^* P^+_{\mathfrak{N}}(I-V)\varphi].
\end{aligned}
$$

This implies

$$A^*(I+U)\tilde{f}_i = i(\tilde{f}_i - U\tilde{f}_i).$$

Hence

$$A^*\left(\frac{t\tilde{e}}{t^2+1}\right) = -\frac{\tilde{e}}{t^2+1}, \quad \tilde{e} \in F_\infty. \tag{89}$$

Let $Q \in [E, E]$ be the operator in the definition of the class $N(R)$. We introduce a new operator R_0 acting in the following way:

$$R_0 f = iQ\tilde{\mathcal{R}}^{-1} A^* P_1 f, \quad f \in \mathfrak{H}_+. \tag{90}$$

In order to show that $R_0 \in [\mathfrak{H}_+, E]$, we consider the following calculation for $f \in \mathfrak{H}_+$:

$$\begin{aligned}\|R_0 f\|_E &= \sup_{g\in E} \frac{|(R_0 f, g)_E|}{\|g\|_E} = \sup_{g\in E} \frac{|(Q\tilde{\mathcal{R}}^{-1}A^*P_1 f, g)_E|}{\|g\|_E} \\ &= \sup_{g\in E} \frac{|(\tilde{\mathcal{R}}^{-1}A^*P_1 f, Qg)_E|}{\|g\|_E} \leq \sup_{g\in E} \frac{\|\tilde{\mathcal{R}}^{-1}A^*P_1 f\|_E \cdot \|Qg\|_E}{\|g\|_E} \\ &\leq c\|A^*P_1 f\|_{\tilde{\mathfrak{H}}_+} \leq b\|A^*P_1 f\|_{\mathfrak{H}_+}, \quad b, c \text{ - constants.}\end{aligned}$$

Here we used that $P_1 f \subset \mathfrak{D}(\hat{A})$, for all $f \in \tilde{\mathfrak{H}}_+$, formulas (65) and (89), and the equivalence of the norms $\|\cdot\|_{\tilde{\mathfrak{H}}_+}$ and $\|\cdot\|_+$.

For $f \in \mathfrak{H}_+$, we have $P_1 f = (I-V)\varphi$, $\varphi \in N$ and

$$A^*P_1 f = i(V+I)\varphi + iA^*P^+_{\mathfrak{N}}(V-I)\varphi.$$

We now have

$$\begin{aligned}\|A^*P^+_{\mathfrak{N}}(V-I)\varphi\|^2_+ &= \|A^*P^+_{\mathfrak{N}}(V-I)\varphi\|^2 + \|A^*A^*P^+_{\mathfrak{N}}(V-I)\varphi\|^2 \\ &= \|A^*P^+_{\mathfrak{N}}(V-I)\varphi\|^2 + \|PP^+_{\mathfrak{N}}(V-I)\varphi\|^2 \\ &\leq \|A^*P^+_{\mathfrak{N}}(V-I)\varphi\|^2 + \|P^+_{\mathfrak{N}}(V-I)\varphi\|^2 \\ &= \|P^+_{\mathfrak{N}}(V-I)\varphi\|^2_+,\end{aligned}$$

and

$$\begin{aligned}\|i(V+I)\varphi + iA^*P^+_{\mathfrak{N}}(V-I)\varphi\|^2_+ &= \|A^*P^+_{\mathfrak{N}}(V-I)\varphi\|^2_+ + \|\varphi + V\varphi\|^2_+ \\ &\leq \|P^+_{\mathfrak{N}}(V-I)\varphi\|^2_+ + \|\varphi + V\varphi\|^2_+ \\ &= \|\varphi - V\varphi\|^2_+.\end{aligned}$$

This implies that there exists a constant k such that

(91) $$\|A^*P_1 f\| \leq \|P_1 f\|_+ \leq k\|f\|_+, \quad \forall f \in \mathfrak{H}_+.$$

Therefore, for some constant $d > 0$ we have $\|R_0 f\|_{\leq} d\|f\|_+$, $\forall f \in \mathfrak{H}_+$. Thus, $R_0 \in [\mathfrak{H}_+, E]$.

Let R_0^* be the adjoint operator to R_0, i.e. $R_0^* \in [E, \mathfrak{H}_-]$ and for all $f \in \mathfrak{H}_+$, $e \in E$, $(R_0 f, e)_E = (f, R_0^* e)$. Since $R_0(\mathfrak{D}(\hat{A})) = 0$, $\mathfrak{R}(R_0^*)$ is $(\cdot)$-orthogonal to $\mathfrak{D}(\hat{A})$. Letting $\mathfrak{M} = \mathfrak{N}'_{-i} \oplus \mathfrak{N}'_i \oplus \mathfrak{N}$, we obtain from (88)

(92) $$\mathfrak{M} = (V+I)(\mathfrak{N}'_i \oplus \mathfrak{N}) \dot{+} (I-V)N.$$

In the space $\mathfrak{M}$ we define an operator S in the following way

(93) $$\begin{aligned}S(\varphi + V\varphi) &= \frac{i}{2}(I-V)\varphi, \quad \varphi \in \mathfrak{N}'_i \oplus \mathfrak{N}, \\ S(\varphi_N - V\varphi_N) &= \left[-\mathcal{R}_1(R_0^* + P_0^*)\tilde{\mathcal{R}}^{-1}A^* + \frac{i}{2}(P^+_{\mathfrak{N}'_i} - P^+_{\mathfrak{N}'_{-i}})\right](\varphi_N - V\varphi_N),\end{aligned}$$

where $\varphi_N \in N$. In order to show that S is a (1)-self-adjoint operator on $\mathfrak{M}$, we first check that

$$(S(\varphi + V\varphi), \varphi + V\varphi)_1 = (\varphi + V\varphi, S(\varphi + V\varphi))_1, \quad \varphi \in \mathfrak{N}'_i \oplus \mathfrak{N}. \tag{94}$$

It is easy to see that

$$(P^+_{\mathfrak{N}'_i} - P^+_{\mathfrak{N}'_{-i}})(\varphi_N - V\varphi_N) = \varphi_N + V\varphi_N, \quad \varphi_N \in N.$$

This follows from the definition of the space N and the fact that φ_N belongs to $\mathfrak{N}'_i$. Furthermore, since $\varphi_N \in \mathfrak{N}'_i$, and $V\varphi_N \in \mathfrak{N}'_{-i}$ we have that $P^+_{\mathfrak{N}'_{-i}}\varphi_N = \varphi_N$, $P^+_{\mathfrak{N}'_{-i}}V\varphi_N = \varphi_N$, and $P^+_{\mathfrak{N}'_i}V\varphi_N = P^+_{\mathfrak{N}'_{-i}}\varphi_N = 0$. Consequently,

$$\begin{aligned} ((\varphi_N + V\varphi_N), \varphi_N - V\varphi_N)_1 &= \|\varphi_N\|_1^2 - \|V\varphi_N\|_1^2 \\ &= \|P^+_{\mathfrak{N}'_i}\varphi_N\|_1^2 - \|P^+_{\mathfrak{N}'_{-i}}V\varphi_N\|_1^2 = 0. \end{aligned} \tag{95}$$

Since $P_0(I - V)N = 0$, we have

$$(\mathcal{R}_1 P_0^* \tilde{\mathcal{R}}^{-1} A^*(\varphi_N - V\varphi_N), \varphi_N - V\varphi_N) = (\tilde{\mathcal{R}}^{-1} A^*(\varphi_N - V\varphi_N), P_0(\varphi_N - V\varphi_N)) = 0. \tag{96}$$

This allows us to consider only the R_0^*-containing part of (93), i.e.

$$\begin{aligned} (S(\varphi_N - V\varphi_N), \varphi_N - V\varphi_N)_1 &= (-\mathcal{R}_1 R_0^* \tilde{\mathcal{R}}^{-1} A^*(\varphi_N - V\varphi_N), (\varphi_N - V\varphi_N)_1 \\ &= (\tilde{\mathcal{R}}^{-1} A^*(\varphi_N - V\varphi_N), -R_0(\varphi_N - V\varphi_N))_E \\ &= (\tilde{\mathcal{R}}^{-1} A^*(\varphi_N - V\varphi_N), iQ\tilde{\mathcal{R}}^{-1} A^* P_1(\varphi_N - V\varphi_N))_E \\ &= (-iQ\tilde{\mathcal{R}}^{-1} A^*(\varphi_N - V\varphi_N), \tilde{\mathcal{R}}^{-1} A^*(\varphi_N - V\varphi_N))_E \\ &= ((\varphi_N - V\varphi_N), R_0^* \mathcal{R}^{-1} A^*(\varphi_N - V\varphi_N))_E \\ &= ((\varphi_N - V\varphi_N), \mathcal{R}_1 R_0^* \mathcal{R}^{-1} A^*(\varphi_N - V\varphi_N))_1 \\ &= ((\varphi_N - V\varphi_N), S(\varphi_N - V\varphi_N))_1. \end{aligned}$$

Now we will show that

$$(S(\varphi + V\varphi), \varphi_N - V\varphi_N)_1 = (\varphi + V\varphi, S(\varphi + V\varphi))_1, \quad \varphi_N \in N, \varphi \in \mathfrak{N}'_i \oplus \mathfrak{N}. \tag{97}$$

Let us note that $P^+_{\mathfrak{N}}(\varphi_N + V\varphi_N) = 0$ implies $P^+_{\mathfrak{N}}\varphi_N = -P^+_{\mathfrak{N}}V\varphi_N$. Also, $(\varphi, \varphi_N)_1 = (V\varphi, V\varphi_N)_1$, since V is a (1)-isometric mapping. We will now show that the orthogonality relations yield $(\varphi, V\varphi_N)_1 = (\varphi, P^+_{\mathfrak{N}} V\varphi_N)_1 = 0$. First we need a calculation

$$\begin{aligned}
(S(\varphi + V\varphi), \varphi_N - V\varphi_N)_1 &= \frac{i}{2}((I - V)\varphi, \varphi_N - V\varphi_N)_1 \\
&= i(\varphi, \varphi_N)_1 - \frac{i}{2}(\varphi, V\varphi_N)_1 - \frac{i}{2}(V\varphi, \varphi_N)_1 \\
&= i(\varphi, \varphi_N)_1 - \frac{i}{2}(\varphi, V\varphi_N)_1 - \frac{i}{2}(\varphi, P^+_{\mathfrak{N}} V\varphi_N)_1 \\
&= i(\varphi, \varphi_N)_1 - \frac{i}{2}(\varphi, V\varphi_N)_1 + \frac{i}{2}(\varphi, P^+_{\mathfrak{N}} V\varphi_N)_1 \\
&= i(\varphi, \varphi_N)_1 + \frac{i}{2}(P^+_{\mathfrak{N}}(I - V)\varphi, \varphi_N)_1.
\end{aligned}$$

Also, note that

$$\left(\varphi + V\varphi, \frac{i}{2}(P^+_{\mathfrak{N}'_i} - P^+_{\mathfrak{N}'_{-i}})(\varphi_N - V\varphi_N)\right)_1 = -\frac{i}{2}\left(\varphi + V\varphi, \varphi_N + V\varphi_N\right)_1,$$

and

$$\begin{aligned}
(\varphi + V\varphi, S(\varphi_N - V\varphi_N)_1 = (\varphi + V\varphi, -\mathcal{R}_1(R_0^* + P_0^*)\tilde{\mathcal{R}}^{-1}A^*(\varphi_N - V\varphi_N))_1 \\
- \frac{i}{2}(\varphi + V\varphi, \varphi_N + V\varphi_N)_1.
\end{aligned}$$

Next, recall that $\mathfrak{R}(R_0^*)$ is $(\cdot)$-orthogonal to $\mathfrak{D}(\hat{A})$ and

$$\varphi + V\varphi \in \mathfrak{D}(\hat{A}) = \mathfrak{D}(A) \oplus (V + I)(\mathfrak{N}'_i \oplus \mathfrak{N}).$$

It follows that

$$\begin{aligned}
(\varphi + V\varphi, \mathcal{R}_1 R_0^* \tilde{\mathcal{R}}^{-1}A^*(\varphi_N - V\varphi_N))_1 &= (\varphi + V\varphi, R_0^*\tilde{\mathcal{R}}^{-1}A^*(\varphi_N - V\varphi_N)) = 0, \\
(\varphi + V\varphi, -\mathcal{R}_1 P_0^* \tilde{\mathcal{R}}^{-1}A^*(\varphi_N - V\varphi_N))_1 &= -(\varphi + V\varphi, A^*(\varphi_N - V\varphi_N))_{\mathfrak{H}_+} \\
&= -(\varphi + V\varphi, A^*(\varphi_N - V\varphi_N)) \\
&\quad - (\hat{A}(\varphi + V\varphi), \tilde{A}_0 A^*(\varphi_N - V\varphi_N)).
\end{aligned}$$

Applying Theorem 1 we obtain:

$$\hat{A}(\varphi + V\varphi) = A^*(\varphi + V\varphi) + \frac{i}{2}\mathcal{R}^{-1}P^+_{\mathfrak{N}}(I - V)\varphi,$$

$$A^*(\varphi_N - V\varphi_N) = i(I + V)\varphi_N + A^* P^+_{\mathfrak{N}}(I - V)\varphi_N,$$

$$\begin{aligned}
\hat{A}A^*(\varphi_N - V\varphi_N) &= AA^* P^+_{\mathfrak{N}}(I - V)\varphi_N + iA^*(V + I)\varphi_N - \frac{i}{2}\mathcal{R}_1^{-1}P^+_{\mathfrak{N}}(I - V)\varphi_N \\
&= iA^*(V + I)\varphi_N - P^+_{\mathfrak{N}}(I - V)\varphi_N.
\end{aligned}$$

Here we used the following relations:

$$A^*(I-V)\in\mathfrak{D}(A),$$

$$\hat{A}(f_i-Uf_i)=A^*(\varphi+V\varphi)+iP^+_{\mathfrak{N}}(I-V)\varphi,$$
$$f_i-Uf_i=\varphi+V\varphi+iA^*P^+_{\mathfrak{N}}(V-I)\varphi,$$
$$\hat{A}(\varphi+V\varphi)=A^*(\varphi+V\varphi)+\frac{i}{2}\mathcal{R}^{-1}P^+_{\mathfrak{N}}(I-V)\varphi,$$

and

$$AA^*P^+_{\mathfrak{N}}(I-V\varphi_N-\frac{1}{2}\mathcal{R}^{-1}(I-V)\varphi=-P^+_{\mathfrak{N}}(I-V)\varphi_N.$$

The above identities yield that

$$(\varphi+V\varphi,A^*(\varphi_N-V\varphi_N))_{\tilde{\mathfrak{H}}_+}=(\varphi+V\varphi,i(\varphi_N+V\varphi_N))_1-i(P^+_{\mathfrak{N}}(I-V)\varphi,\varphi_N)_1.$$

Thus,

$$\begin{aligned}(\varphi+V\varphi,-\mathcal{R}_1P^+_0\tilde{\mathcal{R}}^{-1}A^*(\varphi_N-V\varphi_N))_0&=i(\varphi+V\varphi,\varphi_N+V\varphi_N)\\&\quad+i(P^+_N(I-V)\varphi,\varphi_N),\\(\varphi+V\varphi,\frac{i}{2}(\varphi_N+V\varphi_N))_1&=-\frac{i}{2}(\varphi+V\varphi,\varphi_N+V\varphi_N)_1,\end{aligned}$$

and

$$\begin{aligned}(\varphi+V\varphi,S(\varphi_N-V\varphi_N))_1&=i(\varphi+V\varphi,\varphi_N+V\varphi_N)\\&\quad+i(P^+_{\mathfrak{N}}(I-V)\varphi,\varphi_N)_1-\frac{i}{2}(\varphi+\varphi,\varphi_N+V\varphi_N)_1\\&=i(\varphi,\varphi_N)_1+\frac{i}{2}(V\varphi,\varphi_N)_1\\&\quad+\frac{i}{2}(\varphi,V\varphi_N)_1+i(P^+_{\mathfrak{N}}(I-V)\varphi,\varphi_N)_1\\&=i(\varphi,\varphi_N)_1+\frac{i}{2}(P^+_{\mathfrak{N}}(I-V)\varphi,\varphi_N)_1\\&=(S(\varphi+V\varphi),\varphi_N-V\varphi_N).\end{aligned}$$

This shows that S is a (1)-self-adjoint operator in $\mathfrak{M}$.

By Corollary 2, a self-adjoint bi-extension of the operator A is defined by the formula

$$\mathbb{B}=AP^+_{\mathfrak{D}(A)}+\left[A^*+\mathcal{R}^{-1}\left(S-\frac{i}{2}P^+_{\mathfrak{N}'_i}+\frac{i}{2}P^+_{\mathfrak{N}'_{-i}}\right)\right]P^+_{\mathfrak{M}},\tag{98}$$

where S is defined by (97). Obviously, if $f=f_A+(V+I)\varphi$, $\varphi\in\mathfrak{N}'_i\oplus\mathfrak{N}$, and $f_A\in\mathfrak{D}(A)$ then $\mathbb{B}f=\hat{A}f$. This means that the quasi-kernel of the operator $\mathbb{B}$ coincides with $\hat{A}$.

STEP 4. In this Step we will construct a $(*)$-extension of some operator of the class Λ_A. First, we introduce the bounded linear operator K acting from the space E into the space $\mathfrak{H}_-$ as follows:

$$Ke = (P_0^* + R_0^*)P_{F_\infty} + \hat{I}P_{E_\infty}e, \quad e \in E, \tag{99}$$

where P_{F_∞} and P_{E_∞} are orthogonal projections of the space E onto F_∞ and E_∞ respectively, and $\hat{I}$ is an embedding of E_∞ in $\mathfrak{H}_-$.

Let $K^* \in [\mathfrak{H}_+, E]$ be an adjoint of the operator K, i.e.

$$(Kf, g) = (f, K^*g), \quad f \in E,\, g \in \mathfrak{H}_+.$$

Let

$$\mathbb{C} = K^*JK, \tag{100}$$

where $J \in [E, E]$ satisfies $J = J^* = J^{-1}$. Since $\mathfrak{R}(K)$ is orthogonal to $\mathfrak{D}(A)$, $\mathbb{C}(\mathfrak{D}(A)) = 0$. Moreover, $(\mathbb{C}f, g) = (f, \mathbb{C}g)$ for all $f \in \mathfrak{H}_+$, $g \in \mathfrak{H}_+$.

We define an operator $\mathbb{A}$ by

$$\mathbb{A} = \mathbb{B} + i\mathbb{C}. \tag{101}$$

We now show that $\mathbb{A}$ is a $(*)$-extension of some operator T of the class Λ_A.

Let λ be a regular point of the operator $\hat{A}$ and let $\hat{R}_\lambda = \overline{(\mathbb{B} - \lambda I)^{-1}}$. Also, note that

$$(\hat{R}_\lambda f, g) = (f, (\hat{A} - \bar{\lambda} I)^{-1} g), \quad \forall f \in \mathfrak{H}_-,\, g \in \mathfrak{H}.$$

As it was shown in Step 1 (see (71))

$$(\overline{\hat{A}} - \lambda I)^{-1} = \frac{e}{t - \lambda}, \quad \forall e \in E,$$

where E is considered as a subspace of $\tilde{\mathfrak{H}}_-$. Clearly,

$$\begin{aligned}(\hat{R}_\lambda P_0^* e, g) &= (P_0^* e, (\hat{A} - \bar{\lambda} I)^{-1} g) \\ &= (e, (\hat{A} - \bar{\lambda} I)^{-1} g) = ((\overline{\hat{A}} - \lambda I)^{-1} e, g),\ \forall e \in E,\, g \in \mathfrak{H} = L^2_G(E).\end{aligned}$$

It follows that

$$\hat{R}_\lambda P_0^* e = \frac{e}{t - \lambda}, \quad \forall e \in E. \tag{102}$$

Since $R_0(\mathfrak{D}(\hat{A})) = 0$, $R_0(\hat{A} - \bar{\lambda} I)^{-1} g = 0$, for all $g \in \mathfrak{H}$, and we have

$$(\hat{R}_\lambda R_0^* e, g) = (R_0^* e, (\hat{A} - \bar{\lambda} I)^{-1} g) = (e, R_0(\hat{A} - \bar{\lambda} I)^{-1} g) = 0,$$

$$\hat{R}_\lambda K e_1 = \hat{R}_\lambda P_0^* e_1 = \frac{e_1}{t-\lambda}, \quad e_1 \in F_\infty,$$
$$\hat{R}_\lambda K e_2 = \hat{R}_\lambda e_2 \frac{e_2}{t-\lambda} \in \tilde{\mathfrak{H}}_+, \quad e_2 \in E_\infty,$$

This implies that the operator K is invertible. Indeed, if $Ke = 0$, then $(P_0^* + R_0^*) e_1 = -\hat{I} e_2$ and $\hat{R}_\lambda K e = 0$. Hence, $\hat{R}_\lambda (P_0^* + R_0^*) \tilde{e} = -\hat{R}_\lambda e_2$. That is,

$$\frac{e_1}{t-\lambda} = \frac{e_2}{t-\lambda}, \quad e = \hat{e} + e_1,$$

which implies that $e = 0$.

We should also note that $\hat{R}_\lambda K \in [E, \mathfrak{H}_+]$, since $\hat{R}_\lambda$ maps $\mathfrak{R}(K)$ into $\mathfrak{H}_+$ continuously.

Let us consider now the operator-valued function V defined by

$$V(\lambda) = K^* \hat{R}_\lambda K, \quad \mathrm{Im}\lambda \neq 0. \tag{103}$$

Obviously, $(V(\lambda)e, h)_E = (\hat{R}_\lambda K e, Kh)$ for $e \in E$, $h \in E$, $e = e_1 + e_2$, $h = h_1 + h_2$. Therefore,

$$\begin{aligned}
(\hat{R}_\lambda K e, Kh) &= (\hat{R}_\lambda (P_0^* + R_0^*) e_1 + \hat{R}_\lambda e_2, (P_0^* + R_0^*) h_1 + \hat{I} h_2) \\
&= (\hat{R}_\lambda P_0^* e_1 + \hat{R}_\lambda e_2, (P_0^* + R_0) h_1 + \hat{I} h_2) \\
&= (\hat{R}_\lambda P_0^* e_1, P_0^* h_1) + (\hat{R}_\lambda P_0^* e_1, R_0^* h_1) + (\hat{R}_\lambda P_0^* e_1, h_2) + (\hat{R}_\lambda e_2, P_0^* h_1) \\
&+ (\hat{R}_\lambda e_2, R_0^* h_2) + (\hat{R}_\lambda e_2, h_2) \\
&= (P_0 \hat{R}_\lambda P_0^* e_1, h_1) + (P_0 \hat{R}_\lambda P_0^* e_1, h_2) + (\hat{R}_\lambda P_0^* e_1, h_2) + (\hat{R}_\lambda e_2, h_2) \\
&+ (R_0 \hat{R}_\lambda e_2, h_2)_E + (\hat{R}_\lambda e_2, h_2).
\end{aligned}$$

We also have

$$\hat{R}_\lambda P_0^* e_1 = \frac{e_1}{t-\lambda} \notin \tilde{\mathfrak{H}}_-.$$

Consider an element

$$\frac{e_1}{t-\lambda} - \frac{t e_1}{t^2+1} = -\frac{\lambda t e_1}{(t-\lambda)(t^2+1)}, \quad e_1 \in F_\infty.$$

Clearly

$$\int\limits_{-\infty}^{+\infty} \frac{|\lambda|^2 t^4}{|t-\lambda|^2 (t^2+1)} \cdot \frac{d(G(t)e_1, e_1)_E}{1+t^2} < \infty,$$

and hence

$$\frac{e_1}{t-\lambda} - \frac{te_1}{t^2+1} \in \mathfrak{D}(\hat{A}).$$

Moreover,

$$\frac{te_1}{t^2+1} \in (I-V)N, \quad e_1 \in F_\infty.$$

This implies

$$\begin{aligned} P_0\left\{\frac{e_1}{t-\lambda}\right\} &= \frac{e_1}{t-\lambda} - \frac{te_1}{t^2+1}, \\ P_1\left\{\frac{e_1}{t-\lambda}\right\} &= \frac{te_1}{t^2+1}. \end{aligned}$$

Consequently,

$$(P_0\hat{R}_\lambda P_0^* e_1, h_2) = \int\limits_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\right) d(G(t)e_1, h_2)_E.$$

We also have that

$$(R_0\hat{R}_\lambda P_0^*, h_1)_E = -(Q\tilde{\mathcal{R}}^{-1}A^*P_1\hat{R}_\lambda P_0^* e_1, h_1)_E = -(\tilde{\mathcal{R}}^{-1}A^*P_1\hat{R}_\lambda P_0 e_1, Qh_1)_E.$$

From (65) and (89) we obtain

$$\tilde{\mathcal{R}}^{-1}A^*P_1\hat{R}_\lambda P_0^* e_1 = \tilde{\mathcal{R}}^*\left(\frac{e_1}{t^2+1}\right) = -e_1,$$

from which it follows that

$$(R_0\hat{R}_\lambda P_0^*, h_2)_E = (e_1, Qh_2)_E = (Qe_1, h_2)_E.$$

Furthermore we obtain

$$\begin{aligned} (\hat{R}_\lambda P_0^* e_1, h_2) &= \int\limits_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda}\right) d(G(t)e_2, h_2)_E \\ &= \int\limits_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda}\right) d(G(t)e_2, h_2)_E - (Qe_1, h_2)_E + (Qe_1, h_2)_E \\ &= \int\limits_{-\infty}^{+\infty} \frac{t}{t^2+1} d(G(t)e_1, h_2)_E + (Qe_1, h_2)_E \\ &= \int\limits_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\right) d(G(t)e_1, h_2)_E + (Qe_1, h_2)_E. \end{aligned}$$

Since $R_0\hat{R}_\lambda e_2 = 0$, we have

$$(\hat{R}_\lambda e_2, h_1) = \int\limits_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda}\right) d(G(t)e_2, h_1)_E - (Qe_2, h_1)_E + (Qe_2, h_1)_E$$
$$= \int\limits_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\right) d(G(t)e_2, h_1)_E + (Qe_2, h_1)_E$$

Thus,

$$(\hat{R}_\lambda e_2, h_2) = \int\limits_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\right) d(G(t)e_2, h_2)_E + (Qe_2, h_2)_E \tag{104}$$

These calculations imply

$$(\hat{R}_\lambda e, h) = \int\limits_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\right) d(G(t)e, h)_E + (Qe, h)_E,$$

hence,

$$(V(\lambda)e, h) = \int\limits_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\right) d(G(t)e, h)_E + (Qe, h)_E \tag{105}$$

Next, we show that $(\mathbb{B} + iI)\hat{R}_{\pm i}Ke = Ke$, for all $e \in E$, where $\mathbb{B}$ is the strong self-adjoint bi-extension defined by (98). By Theorem 7, the equation $(\mathbb{B} - \lambda I)x = f$ has a unique solution x for any

$$f \in \Re\left[\mathcal{R}_1^{-1}\left(S - \frac{i}{2}P^+_{\mathfrak{N}'_i} + \frac{i}{2}P^+_{\mathfrak{N}'_{-i}}\right)\right] + E_\infty.$$

We will now show that in fact

$$\Re(K) = \Re\left[\mathcal{R}_1^{-1}\left(S - \frac{i}{2}P^+_{\mathfrak{N}'_i} + \frac{i}{2}P^+_{\mathfrak{N}'_{-i}}\right)\right] + E_\infty.$$

If $\varphi_N \in N$, then

$$\left(S - \frac{i}{2}P^+_{\mathfrak{N}'_i} + \frac{i}{2}P^+_{\mathfrak{N}'_{-i}}\right)(\varphi_N - V\varphi_N) = \mathcal{R}_1(R_0^* + P_{)}^*)\tilde{\mathcal{R}}^{-1}A^*(\varphi_N - V\varphi_N).$$

Using (89) we can conclude that $\tilde{\mathcal{R}}^{-1}(I-V)N = F_\infty$, and hence

$$\mathfrak{R}\left[\mathcal{R}_1^{-1}\left(S-\frac{i}{2}P^+_{\mathfrak{N}'_i}+\frac{i}{2}P^+_{\mathfrak{N}'_{-i}}\right)\right](I-V)N = (P_0^*+R_0^*)F_\infty.$$

Letting $P^+ = P^+_{\mathfrak{N}'_i} + P^+_{\mathfrak{N}'_{-i}}$, we have

$$P^+\left(S-\frac{i}{2}P^+_{\mathfrak{N}'_i}+\frac{i}{2}P^+_{\mathfrak{N}'_{-i}}\right)(I+V)\varphi = 0,\ \varphi\in\mathfrak{M}.$$

Therefore,

$$E_\infty + \mathfrak{R}\left[\tilde{\mathcal{R}}^{-1}\left(S-\frac{i}{2}P^+_{\mathfrak{N}'_i}+\frac{i}{2}P^+_{\mathfrak{N}'_{-i}}\right)\right] = \mathfrak{R}(K).$$

Since $\hat{R}_\lambda = \overline{(\mathbb{B}-\lambda I)^{-1}}$, the above calculations imply

$$(\mathbb{B}-\lambda I)^{-1}Ke = \hat{R}_\lambda Ke, \tag{106}$$

for all $e\in E$. For $\mathrm{Im}\lambda\neq 0$ we have that $\hat{R}_\lambda KE = \mathfrak{N}_\lambda$ is the defect space of the operator A. Therefore $(\mathbb{B}+iI)\hat{R}_{\pm i}Ke = Ke$ and $\hat{R}_{\pm i}KE = \mathfrak{N}_{\pm i}$.

Taking into account (105) we get

$$\begin{aligned} V(-i) &= \int_{-\infty}^{+\infty}\left(\frac{1}{t+i}-\frac{t}{t^2+1}\right)dG(t)+Q \\ &= -i\int_{-\infty}^{+\infty}\frac{dG(t)}{1+t^2}+Q \\ &= -iB+Q. \end{aligned} \tag{107}$$

Therefore,

$$iV(-i)J + I = BJ + iQJ + I. \tag{108}$$

The operator $iV(-i)J+I$ is invertible and so is the right hand side of (108). Since $I+BJ+iQJ = J(I+JB+iJQ)J$, where J is a unitary self-adjoint operator in the space E, 0 is a regular point for the operator $I+BJ+iJQ$. At the same time 0 is a regular point for the operators $I+JB-iJQ = (BJ+iQJ+I)^*$ and $I+BJ-iQJ = (I+JB+iJQ)^*$. Let

$$\begin{aligned} \mathbb{Z} &= (I+BJ-iQJ)^{-1}, \quad \mathbb{Z}\in[E,E], \\ \mathbb{Z}^* &= (I+JB+iJQ)^{-1}, \quad \mathbb{Z}^*\in[E,E], \end{aligned} \tag{109}$$

and let $\Gamma = (I + JB + iJQ)^{-1}$. Clearly $\mathrm{Ker}\Gamma = 0$. We will show that for any $f \in E$, the equation

$$(\mathbb{A} + iI)g = Kf, \tag{110}$$

has a unique solution $g = \hat{R}_{-i}K\Gamma f$, where $\hat{R}_{-i} = \overline{(\mathbb{B} + iI)^{-1}}$ and $\mathbb{A} = \mathbb{B} + i\mathbb{C}$. Moreover,

$$\mathbb{A}\hat{R}_{-i}K\Gamma f = \mathbb{B}\,\hat{R}_{-i}K\Gamma f + iKJK^*\hat{R}_{-i}K\Gamma f,\ f \in E.$$

As shown above (see also [2])

$$\begin{aligned} K^*\hat{R}_{-i}\Gamma f &= V(-i)\Gamma f = (Q - iB)\Gamma f, \\ iKJK^*\hat{R}_{-i}K\Gamma f &= K(JB + iJQ)\Gamma f \\ &= K(I + JB + iJQ)(I + JB + iJQ)^{-1}f - K\Gamma f \\ &= Kf - K\Gamma f, \quad f \in E. \end{aligned}$$

Also,

$$\begin{aligned} (\mathbb{A} + iI)\hat{R}_{-i}K\Gamma f &= (\mathbb{B} + iI)\hat{R}_{-i}K\Gamma f + iKJK^*\hat{R}_{-i}K\Gamma f \\ &= Kf, \quad f \in E. \end{aligned}$$

If there exists a $g \in \mathfrak{H}_+$ such that $\mathbb{A}g = -ig$, then $g \in \mathfrak{N}_{-i}$. Since $\mathfrak{R}(\Gamma) = E$, we find that $\hat{R}_{-i}K\Gamma E = \mathfrak{N}_{-i}$. Therefore $g = \hat{R}_{-i}K\Gamma e$, $e \in E$, and $(\mathbb{A} + iI)\hat{R}_{-i}K\Gamma e = 0$, $Ke = 0$, $e = 0$, and $g = 0$. It follows that the equation $(\mathbb{A} + iI)g = Kf$ has a unique solution given by $g = \hat{R}_{-i}K\Gamma f$ and $(\mathbb{A} + iI)^{-1}KE = \mathfrak{N}_{-i}$.

Similarly, 0 is the regular point for the operator $I + JB - iJQ$ in E. Let

$$\Gamma_1 = (I + JB - iJQ)^{-1}. \tag{111}$$

In the same way as above, we can show that the equation $(\mathbb{A}^* - iI)gKf$, $f \in E$, has a unique solution of the form $g = \hat{R}_iK\Gamma_1 f$ and $(\mathbb{A}^* - iI)^{-1}KE = \mathfrak{N}_i$.

If $f_i \in \mathfrak{N}_i$, then $f_i = f_A + f_{\mathfrak{M}}$, where $f_A \in \mathfrak{D}(A)$, $f_{\mathfrak{M}} \in \mathfrak{M} = \mathfrak{N}'_i \oplus \mathfrak{N}'_{-i} \oplus \mathfrak{N}$. Therefore,

$$\begin{aligned} A^*f_i &= PAf_A + A^*f_{\mathfrak{M}} = iPf_i, \\ A^*f_{\mathfrak{M}} &= iPf_i - PAf_A, \end{aligned}$$

and

$$\begin{aligned} (\mathbb{A} + iI)f_i &= (A + iI)f_A + iPf_i - PAf_A + if_{\mathfrak{M}} \\ &+ \mathcal{R}_1^{-1}\left(S - \frac{i}{2}P^+_{\mathfrak{N}'_i} + \frac{i}{2}P^+_{\mathfrak{N}'_{-i}}\right)f_{\mathfrak{M}} + iKJK^*f_i, \\ &= (I - P)(A + iI)f_A + i(P - I)f_i \in E_\infty \subset \mathfrak{R}(K). \end{aligned}$$

This implies that

$$(\mathbb{A}+iI)f_i - 2if_i = (\mathbb{A}+iI)f_i.$$

That is $2if_i = (\mathbb{A}+iI)(f_i - f_{-i})$, $(f_{-i} \in \mathfrak{N}_{-i})$. Hence $(\mathbb{A}+iI)\mathfrak{H}_+ \subset \mathfrak{N}_i$. Since

$$(\mathbb{A}+iI)\mathfrak{D}(A) = (A+iI)\mathfrak{D}(A),$$

and $(A+iI)\mathfrak{D}(A) \oplus \mathfrak{N}_i = \mathfrak{H}$, we have $(\mathbb{A}+iI)\mathfrak{H}_+ \subset \mathfrak{H}$. Similarly, $(\mathbb{A}^* - iI)\mathfrak{H}_+ \subset \mathfrak{H}$. Therefore we can conclude that the operators $(\mathbb{A}+iI)^{-1}$ and $(\mathbb{A}^*-iI)^{-1}$ are $(-,\cdot)$-continuous (see [25]). Let

$$\begin{aligned} \mathfrak{D}(T) &= (\mathbb{A}+iI)^{-1}\mathfrak{H}, \\ \mathfrak{D}(T_1) &= (\mathbb{A}^*-iI)^{-1}\mathfrak{H}. \end{aligned} \tag{112}$$

It is easy to see that $\mathfrak{D}(T)$ and $\mathfrak{D}(T_1)$ are dense in $\mathfrak{H}$ and that the operators $(\mathbb{A}+iI)^{-1}\Big|_{\mathfrak{H}}$ and $(\mathbb{A}^*-iI)^{-1}\Big|_{\mathfrak{H}}$ are $(\cdot,\cdot)$-continuous.

Let us define

$$\begin{aligned} T &= \mathbb{A}\Big|_{\mathfrak{D}(T)}, \\ T_1 &= \mathbb{A}^*\Big|_{\mathfrak{D}(T_1)}. \end{aligned} \tag{113}$$

The points (i) and $(-i)$ are regular points for the operators T and T_1 respectively. This implies that $T_1 = T^*$.

Since T and T^* are quasi-kernels of operators $\mathbb{A}$ and $\mathbb{A}^*$ respectively, and $\mathrm{Re}\mathbb{A} = \mathbb{B}$ is a strong self-adjoint bi-extension of the operator A we find that $T \in \Lambda_A$ (the fact that PT and PT^* are closed follows from the $(+,\cdot)$-continuity of T and T^*).

STEP 5. Let us construct a linear stationary conservative dynamical system θ. Let $K \in [E, \mathfrak{H}_-]$ be the operator defined in the Step 4. It is easy to see that

$$\frac{1}{2i}(\mathbb{A}-\mathbb{A}^*) = KJK^*.$$

Therefore,

$$\theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & E \end{pmatrix}$$

is a l.s.c.d.s. In particular, θ is a scattering system if $J = I$. Since $V_\theta(z)$ is a linear-fractional transformation of $W_\theta(z)$ then $V_\theta(z) = V(z)$ whenever z is in some neighborhood G_{-i} of the point $(-i)$. This completes the proof of the theorem.

Remark. It can be seen that when $J = I$ the invertibility condition for $I + iV(\lambda)J$ is satisfied automatically.

Theorem 10. *Let an operator-valued function $V(z)$ belong to the class $N(R)$. Then $V(z)$ can be realized by the scattering ($J = I$) system (dissipative operator colligation) θ of the form (30).*

The following theorem deals with the realization of two realizable operator-valued R-functions differing from each other only by the constant terms in the representation (48).

Theorem 11. *Let the operator-valued functions*

$$V_1(\lambda) = Q_1 + \int_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) dG(t) \tag{114}$$

and

$$V_2(\lambda) = Q_2 + \int_{-\infty}^{+\infty} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) dG(t) \tag{115}$$

belong to the class $N(R)$. Then they can be realized by systems

$$\theta_1 = \begin{pmatrix} \mathbb{A}_1 & K_1 & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & E \end{pmatrix} \qquad (\mathbb{A}_1 \supset T_1) \tag{116}$$

and

$$\theta_2 = \begin{pmatrix} \mathbb{A}_2 & K_2 & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & E \end{pmatrix} \qquad (\mathbb{A}_2 \supset T_2) \tag{117}$$

respectively, so that the operators T_1 and T_2 acting on the Hilbert space $\mathfrak{H}$ are both extensions of the Hermitian operator A defined in this Hilbert space.

Proof. Applying Theorem 9 to the function $V_1(\lambda)$, we obtain a l.s.c.d.s. θ_1 of the type (116). The corresponding Hermitian operator A_1 constructed in the Steps 1 and 2 of the proof of Theorem 9 satisfies the formulas (72) and (73). The construction of A_1 doesn't involve the operator Q_1 from (114). It is easy to see that the corresponding rigged Hilbert space $\mathfrak{H}_+^{(1)} \subset \mathfrak{H}^{(1)} \subset \mathfrak{H}_-^{(1)}$ was built without the use of the operator Q_1 too.

Similarly, if we apply Theorem 9 to the function $V_2(\lambda)$ we get the corresponding Hermitian operator $A_2 = A_1$ and the same rigged Hilbert space. This occurs because the operator-functions $V_1(\lambda)$ and $V_2(\lambda)$ differ from each other only by the constant terms Q_1 and Q_2. Setting $A = A_1 = A_2$, we can conclude that T_1 and T_2 are both extensions of the Hermitian operator A.

A closed Hermitian operator A is called a *prime operator* [25] if there exists no reducing invariant subspace on which it induces a self-adjoint operator.

Definition. A l.s.c.d.s. θ of the form (30) is said to be a *prime system* if its Hermitian operator A is a prime operator.

Theorem 12. *Let the operator-valued function $V(z)$ belong to the class $N(R)$. Then it can be realized by the prime system θ of the form (30) with a preassigned direction operator J for which $I + iV(-i)J$ is invertible.*

Proof. Theorem 9 provides us with a possibility of realization for a given operator-valued function $V(z)$ from the class $N(R)$. Let us assume that its Hermitian operator A has a reducing invariant subspace $\mathfrak{H}^1 \subset \mathfrak{H}$ on which it generates the self-adjoint operator A_1. Then we can write the following $(\cdot)$-orthogonal decomposition

$$\mathfrak{H} = \mathfrak{H}^0 + \mathfrak{H}^1, \qquad A = A_0 \oplus A_1, \tag{118}$$

where A_0 is an operator induced by A on $\mathfrak{H}^0$.

Now let us consider an operator $T \supset A$ as in the definition of the system θ. We have

$$T = T_0 \oplus A_1, \tag{119}$$

where $T_0 \supset A_0$. Indeed, since A_1 is a self-adjoint operator it can not be extended any further. Clearly, $\overline{\mathfrak{D}(A_1)} = \mathfrak{H}^1$. Similarly,

$$T^* = T_0^* \oplus A_1, \tag{120}$$

where $T_0^* \supset A_0$. Furthermore,

$$\mathfrak{H}_+ = \mathfrak{H}_+^0 \oplus \mathfrak{H}_+^1 = \mathfrak{D}(A_0^*) \oplus \mathfrak{D}(A_1).$$

We now show that the same holds in the $(+)$-orthogonality sense. Indeed, if $f_0 \in \mathfrak{H}_+^0$, $f_1 \in \mathfrak{H}_+^1 = \mathfrak{D}(A_1)$ then

$$\begin{aligned}(f_0, f_1)_+ &= (f_0, f_1) + (A^* f_0, A^* f_1)\\ &= (f_0, f_1) + (A_0^* f_0, A_1 f_1)\\ &= 0 + 0 = 0.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- &= \mathfrak{H}_+^0 \oplus \mathfrak{H}_+^1 \subset \mathfrak{H}^0 \oplus \mathfrak{H}^1 \subset \mathfrak{H}_-^0 \oplus \mathfrak{H}_-^1\\ &= \mathfrak{H}_+^0 \oplus \mathfrak{D}(A_1) \subset \mathfrak{H}^0 \oplus \overline{\mathfrak{D}(A_1)} \subset \mathfrak{H}_-^0 \oplus \mathfrak{H}_-^1.\end{aligned}$$

Similarly, we obtain $\mathbb{A} = \mathbb{A}_0 \oplus A_1$ and $\mathbb{A}^* = A_0 \oplus A_1$. Therefore,

$$\begin{aligned}\frac{\mathbb{A} - \mathbb{A}^*}{2i} &= \frac{(\mathbb{A}_0 \oplus A_1) - (\mathbb{A}_0^* \oplus A_1)}{2i}\\ &= \frac{\mathbb{A}_0 - \mathbb{A}_0^*}{2i} \oplus \frac{A_1 - A_1}{2i}\\ &= \frac{\mathbb{A}_0 - \mathbb{A}_0^*}{2i} \oplus O,\end{aligned}$$

where O is the zero operator. This implies that

$$KJK^* = K_0 J K_0^* \oplus O.$$

Let P_+^0 be an orthoprojection operator of $\mathfrak{H}_+$ onto $\mathfrak{H}_+^0$ and set $K = K_0$. Now $K^* = K_0^* P_+^0$, since for all $f \in E$, $g \in \mathfrak{H}_+$ we have:

$$\begin{aligned}(Kf, g) &= (K_0 f, g) = (K_0 f, g_0 + g_1) = (K_0 f, g_0) + (K_0 f, g_1) \\ &= (K_0 f, g_0) = (f, K_0^* g_0) = (f, K_0^* P_+^0 g).\end{aligned}$$

Next, consider $e \in E$ and $x = x^0 + x^1$ in $\mathfrak{H}_+$ such that

$$(\mathbb{A} - \lambda I) P_+^0 x = Ke.$$

Then

$$\begin{aligned}(\mathbb{A}_0 \oplus A_1 - \lambda I) P_+^0 x &= K_0 e, \\ \mathbb{A}_0 x^0 - \lambda x^0 &= K_0 e, \\ (\mathbb{A} - \lambda I) x^0 &= K_0 e, \\ x^0 &= (\mathbb{A}_0 - \lambda I)^{-1} K_0 e.\end{aligned}$$

On the other hand, $x^0 = (\mathbb{A} - \lambda I)^{-1} Ke$. Therefore

$$(\mathbb{A} - \lambda I)^{-1} Ke = (\mathbb{A}_0 - \lambda I)^{-1} K_0 e,$$

and

$$K^* (\mathbb{A} - \lambda I)^{-1} Ke = K_0^* (\mathbb{A}_0 - \lambda I)^{-1} K_0 e.$$

This means that the transfer operator-functions of our system θ and of the system

$$\theta_0 = \begin{pmatrix} \mathbb{A}_0 & K_0 & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & E \end{pmatrix}$$

coincide. This proves the statement of the theorem.

4. Example

Let

$$Tx = \frac{1}{i} \frac{dx}{dt},$$

with

$$\mathfrak{D}(T) = \left\{ x(t) : \, x'(t) \in L^2_{[0,l]}, x(0) = 0 \right\},$$

be a differential operator in $\mathfrak{H} = L^2_{[0,l]}$ $(l > 0)$. Obviously,

$$T^*x = \frac{1}{i}\frac{dx}{dt},$$

with

$$\mathfrak{D}(T^*) = \left\{x(t) : x'(t) \in L^2_{[0,l]}, x(l) = 0\right\},$$

is the adjoint operator of T. Consider the Hermitian operator A (see also [1]) defined by

$$\begin{aligned} Ax &= \frac{1}{i}\frac{dx}{dt}, \\ \mathfrak{D}(A) &= \left\{x(t) : x'(t) \in L^2_{[0,l]}, x(0) = x(l) = 0\right\}, \end{aligned}$$

where its adjoint A^* is given by

$$\begin{aligned} A^*x &= \frac{1}{i}\frac{dx}{dt}, \\ \mathfrak{D}(A^*) &= \left\{x(t) : x'(t) \in L^2_{[0,l]}\right\}. \end{aligned}$$

Then $\mathfrak{H}_+ = \mathfrak{D}(A^*) = W_2^1$ is a Sobolev space with scalar product

$$(x, y)_+ = \int_0^l x(t)\overline{y(t)}\,dt + \int_0^l x'(t)\overline{y'(t)}\,dt.$$

We construct the rigged Hilbert space [9]

$$W_2^1 \subset L^2_{[0,l]} \subset (W_2^1)_-,$$

and consider the operators

$$\begin{aligned} \mathbb{A}x &= \frac{1}{i}\frac{dx}{dt} + ix(0)\left[\delta(x-l) - \delta(x)\right], \\ \mathbb{A}^*x &= \frac{1}{i}\frac{dx}{dt} + ix(l)\left[\delta(x-l) - \delta(x)\right], \end{aligned}$$

where $x(t) \in W_2^1$, $\delta(x)$, $\delta(x-l)$ are delta-functions in $(W_2^1)_-$. It is easy to see that

$$\mathbb{A} \supset T \supset A, \qquad \mathbb{A}^* \supset T^* \supset A,$$

and

$$\theta = \begin{pmatrix} \frac{1}{i}\frac{dx}{dt} + ix(0)[\delta(x-l) - \delta(x)] & K & -1 \\ & & \\ W_1^2 \subset L^2_{[0,l]} \subset (W_2^1)_- & & \mathbb{C}^1 \end{pmatrix} \quad (J = -1)$$

is a Brodskiĭ-Livšic rigged operator colligation where

$$Kc = c \cdot \frac{1}{\sqrt{2}}[\delta(x-l) - \delta(x)], \quad (c \in \mathbb{C}^1)$$
$$K^*x = \left(x, \frac{1}{\sqrt{2}}[\delta(x-l) - \delta(x)]\right) = \frac{1}{\sqrt{2}}[x(l) - x(0)],$$

for $x(t) \in W_2^1$. Also

$$\frac{\mathbb{A} - \mathbb{A}^*}{2i} = -\left(\cdot, \frac{1}{\sqrt{2}}[\delta(x-l) - \delta(x)]\right) \frac{1}{\sqrt{2}}[\delta(x-l) - \delta(x)].$$

The characteristic function of this colligation is

$$W_\theta(\lambda) = I - 2iK^*(\mathbb{A} - \lambda I)^{-1}KJ = e^{i\lambda l}.$$

Consider the following R-function (hyperbolic tangent)

$$V(\lambda) = -i \tanh\left(\frac{i}{2}\lambda l\right).$$

Obviously this fucntion can be realized as follows

$$V(\lambda) = -i \tanh\left(\frac{i}{2}\lambda l\right) = -i\frac{e^{\frac{i}{2}\lambda l} - e^{-\frac{i}{2}\lambda l}}{e^{\frac{i}{2}\lambda l} + e^{-\frac{i}{2}\lambda l}} = -i\frac{e^{i\lambda l} - 1}{e^{i\lambda l} + 1}$$
$$= i\left[W_\theta(\lambda) + I\right]^{-1}\left[W_\theta(\lambda) - I\right] J. \qquad (J = -1)$$

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Department of Mathematics
Troy State University
Troy, AL 36082
E-mail address: sbelyi@trojan.troyst.edu

Department of Mathematics
University of Missouri-Columbia
Columbia, MO 65211
E-mail address: tsekanov@math.missouri.edu

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