REALIZATION AND FACTORIZATION PROBLEMS
FOR J-CONTRACTIVE OPERATOR-VALUED
FUNCTIONS IN HALF-PLANE AND
SYSTEMS WITH UNBOUNDED OPERATORS

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In this paper realization problems for operator-valued $R$-functions acting in finite-dimensional Hilbert space $E$ as linear-fractional transformations of the transfer operator-functions of linear stationary conservative dynamic systems (l.s.c.d.s.) $\theta$ of the form

$$
\begin{cases}
(\mathbb{A} - zI) = KJ\phi_- \\
\phi_+ = \phi_- - 2iK^*x
\end{cases}
$$

are investigated. In a system $\theta$ of the form (0) an operator $\mathbb{A}$ is a bounded linear operator, acting from $\mathcal{H}_+$ into $\mathcal{H}_-$, $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is rigged Hilbert space,

$$
\mathbb{A} \supset T \supset A, \mathbb{A}^* \supset T^* \supset A,
$$

where $A$ is Hermitian operator in $\mathcal{H}, T$ is nonhermitian operator in $\mathcal{H}, K$ is a linear bounded operator from $E$ into $\mathcal{H}, J = J^* = J^{-1}$ and this operator is acting in $E, \phi_\pm \in E, \phi_- \text{ is an input vector, } \phi_+ \text{ is an output vector, } x \in \mathcal{H}_+$ is a vector of an inner state of the system $\theta,$ an operator-valued function

$$
W_\theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ \quad (\phi_+ = W_\theta(z)\phi_-)
$$

is a transfer operator-function of the system $\theta$.

It turns out, that not all operator-valued $R$-functions can be realized in the above mentioned sense and we give a criteria of such a realizability in this paper. In terms of realizable operator-valued $R$-functions we specialize in subclasses of the following types:

1. a subclass for which $\mathcal{D}(A) = \mathcal{H}, \mathcal{D}(T) \neq \mathcal{D}(T^*)$
2. a subclass for which $\mathcal{D}(A) \neq \mathcal{H}, \mathcal{D}(T) = \mathcal{D}(T^*)$
3. a subclass for which $\mathcal{D}(A) \neq \mathcal{H}, \mathcal{D}(T) \neq \mathcal{D}(T^*)$
Given classes of operator-valued $R$-functions allow us to define classes of $J$-contractive operator-valued functions in half-plane, which can be realized as a transfer mapping of the system $\theta$ with, generally speaking, unbounded main operator. A problem when the product of $J$-contractive operator-valued functions from defined classes belongs to the same classes, is investigated.

We consider also a problem of factorization of a realizable $J$-contractive operator-function in half-plane which is connected with invariant subspaces of the main operator $T$ of a system $\theta$.

A class of a realizable $J$-contractive operator-valued functions in half-plane for which main operators $T$ of systems $\theta$ have a property 1) turned out to be very interesting. The theorem on constant $J$-unitary factor in which we show that product (in any order) of an arbitrary constant $J$-unitary operator and $J$-contractive operator-valued function in a half-plane from the mentioned above class belongs to this class, takes place.

We investigate also a problem of connection between realizations of two transfer mappings differing in the constant $J$-unitary factor.

Note that for the first time the problem of studying oscillations in lengthy line with the aid of system theory with unbounded operators had been formulated by M.S. Livšic [11] and later but independently - by J.W. Helton [7].

2.

Let $A$ be a linear closed Hermitian operator, acting in Hilbert space $\mathcal{H}$ with, generally speaking, non-dense domain $\mathcal{D}(A)$. Let $\mathcal{H}_0 = \overline{\mathcal{D}(A)}$, $A^*$-conjugate to $A$ (we consider operator $A$ as acting from $\mathcal{H}_0$ into $\mathcal{H}$). Let us denote $\mathcal{H}_+ = \mathcal{D}(A^*)$ ($\mathcal{D}(A^*) = \mathcal{H}$) and define in $\mathcal{H}_+$ scalar product

$$(f, g)_+ = (f, g) + (A^* f, A^* g) \quad (f, g \in \mathcal{H}_+)$$

and then build the rigged Hilbert space $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$. See [1,2]. We call an operator $A$ regular, if $PA$ is a closed operator in $\mathcal{H}_0$ ($P$ is an orthoprojector $\mathcal{H}$ onto $\mathcal{H}_0$). A regular operator $A$ is called $O$-operator if its semidefect numbers (defect numbers of an operator $PA$) are equal to zero.

An operator $\bar{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ ([\mathcal{H}_+, \mathcal{H}_-] - the set of all linear bounded operators acting from $\mathcal{H}_+$ into $\mathcal{H}_-$) is called biextension of a regular Hermitian operator $A$, if

$$\bar{A} \supset A, \quad A^* \supset A$$

If $\bar{A} = A^*$, then $\bar{A}$ is called a selfadjoint biextension of an operator $A$. Note, that identifying the space conjugate to $\mathcal{H}_{\pm}$ with $\mathcal{H}_{\pm}$. We have that $\bar{A}^* \in [\mathcal{H}_{+}, \mathcal{H}_{-}]$.

We say that the closed linear operator $T$ with dense domain in $\mathcal{H}$ belongs to the class $\Lambda_A$ if:

1. $T \supset A, \quad T^* \supset A$ where $A$ is a maximal common Hermitian part of $T$ and $T^*$ and operator $A$ is regular.
2. $(-i)$ is a regular point of $T$.\textsuperscript{1}

\textsuperscript{1}The condition, that $(-i)$ is a regular point in the definition of the class $\Lambda_A$ is non-essential. It is sufficient to require the existence of some regular point for $T$.
An operator $\mathcal{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a $(\ast)$-extension of an operator $T$ of the class $\Lambda_A$ if
\[ \mathcal{A} \supset T \supset A \]
\[ \mathcal{A}^* \supset T^* \supset A \]
This $(\ast)$-extension $\mathcal{A}$ of an operator $T$ is called correct, if an operator
\[ \mathcal{A}_R = \frac{1}{2}(\mathcal{A} + \mathcal{A}^*) \]
has the following property: An operator
\[ \hat{\mathcal{A}}f = \mathcal{A}_Rf \]
\[ \mathcal{D}(\hat{\mathcal{A}}) = \{ f \in \mathcal{H}_+ : \mathcal{A}_Rf \in \mathcal{H} \} \]
is a selfadjoint operator in Hilbert space $\mathcal{H}$.

**Definition 1.** The aggregate

(1)
\[ \theta = \begin{pmatrix} \mathcal{A} & K \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & J \\ E \end{pmatrix} \]
is called a linear stationary conservative dynamic system if

1. $\mathcal{A}$ is a correct $(\ast)$-extension of an operator $T$ of the class $\Lambda_A$.
2. $J = J^* = J^{-1} \in [E, E]$, \ $\dim E < \infty$
3. $\mathcal{A} - \mathcal{A}^* = 2iJKK^*$, where $K \in [E, \mathcal{H}_-]$ \ ($K^* \in [\mathcal{H}_+, E]$)

In addition an operator $K$ is called a channel operator and $J$ is called a direction operator. System $\theta$ of the form (1) will be called a passage system if $J \neq I$ and a scattering system if $J = I$.

3.

As it is known [8] an operator-function $V(z) \in [E, E]$ is called an operator-valued $R$-function if it is holomorphic in the upper half-plane and $\text{Im} \ V(z) \geq 0$ when $\text{Im} \ z > 0$.

An operator-valued $R$-function, acting in Hilbert space $E(\dim E < \infty)$ has, as it is known [8], integral representation

(2)
\[ V(z) = Q + F \cdot z + \int_{-\infty}^{+\infty} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) dG(t) \]
where $Q = Q^*$, $F \geq 0$ in the Hilbert space $E$, $G(t)$ is non-decreasing operator-function on $(-\infty, +\infty)$ for which
\[ \int_{-\infty}^{+\infty} \frac{dG(t)}{1 + t^2} < \infty. \]
Definition 2. We call an operator-valued \( R \)-function acting in Hilbert space \( E \) \((\dim E < \infty)\) realizable if in some neighbourhood of point \((-i)\) \( V(z) \) can be represented in the form

\[
V(z) = i(W_\theta(z) + I)^{-1}[W_\theta(z) - I]J
\]

where \( W_\theta(z) \) is a transfer operator-function of some l.s.c.d. \( \theta \) with the direction operator \( J \) \((J = J^* = J^{-1} \in [E, E])\).

It may be shown, that the transfer operator-function of the system \( \theta \) of the form (1) has the following properties:

\[
\begin{align*}
W_\theta^*(z)JW_\theta(z) - J &\geq 0 \quad (\Im z > 0, z \in \rho(T)) \\
W_\theta^*(z)JW_\theta(z) - J &= 0 \quad (\Im z = 0, z \in \rho(T)) \\
W_\theta^*(z)JW_\theta(z) - J &\leq 0 \quad (\Im z < 0, z \in \rho(T))
\end{align*}
\]

where \( \rho(T) \) is the set of regular points of an operator \( T \).

Similar relations take place if we change \( W_\theta(z) \) on to \( W_\theta^*(z) \) in (4). Thus, a transfer operator-function of the system \( \theta \) of the form (1) is \( J \)-contractive in the lower half-plane on the set of regular points of an operator \( T \) and \( J \)-unitary on real regular points of an operator \( T \). Let \( \theta \) be a l.s.c.d.s of the form (1). We consider an operator-function

\[
V_\theta(z) = K^*(h_R - zI)^{-1}K
\]

The transfer operator-function \( W_\theta(z) \) of the system \( \theta \) and an operator-function \( V_\theta(z) \) of the form (5) are connected with relation

\[
V_\theta(z) = i(W_\theta(z) + I)^{-1}[W_\theta(z) - I]J
\]

Operator \( T \) of the class \( \Lambda_A \) is called completely nonselfadjoint [4], [10] if there exists no reducing invariant subspace on which one induces a self-adjoint operator.

Realization of an operator-valued \( R \)-function \( V(z) \in [E, E] \) by the system \( \theta \) of the form (1) is called minimal if an operator \( T \) is completely nonselfadjoint.

Definition 3. An operator-valued \( R \)-function \( V(z) \in [E, E] \) \((\dim E < \infty)\) will be said to be a member of the class \( N(R) \) if in the representation (2)

\[
\begin{align*}
i) \quad F &= 0, \\
ii) \quad Qe &= \int_{-\infty}^{+\infty} \frac{dG(t)}{1 + t^2} e
\end{align*}
\]

for all \( e \in E \), when

\[
\int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty
\]

Theorem 1. Let \( \theta \) be a l.s.c.d.s. of the form (1), \( \dim E < \infty \). Then operator-function \( V_\theta(z) \) of the form (5), (6) belongs to the class \( N(R) \). Conversely, let operator-function \( V(z) \) act in a finitedimensional Hilbert space \( E \) and belong to the class \( N(R) \). Then it admits minimal realization by the system \( \theta \) of the form (1) with a preassigned direction operator \( J \) \((J = J^* = J^{-1} \in [E, E])\).

From here, in particular, it follows, that every operator-function of the class \( N(R) \) acting in finitedimensional Hilbert space admits minimal realization with the help of some scattering system \( \theta \) of the form (1).
**Definition 4.** An operator-valued $R$-function $V(z) \in [E, E]$ ($\dim E < \infty$) of the class $N(R)$ is said to be a member of the subclass $N_0(R)$ if in the representation (2)

$$
\int_{-\infty}^{+\infty} (dG(t)e, e)_E = \infty \quad (e \in E, e \not= 0)
$$

From here it follows, that operator-function $V(z)$ of the class $N_0(R)$ has a representation

$$
V(z) = Q + \int_{-\infty}^{+\infty} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) dG(t) \quad (Q = Q^*)
$$

Note, that an operator $Q$ can be an arbitrary self-adjoint operator in Hilbert space $E$.

**Definition 5.** An operator-valued $R$-function $V(z) \in [E, E]$ ($\dim E < \infty$) of the class $N(R)$ is said to be a member of the subclass $N_1(R)$ if in the representation (2)

$$
\int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty \quad (e \in E)
$$

Thus, an operator-function $V(z)$ of the class $N_1(R)$ has a representation

$$
V(z) = \int_{-\infty}^{+\infty} \frac{1}{t-z} dG(t)
$$

An operator-valued $R$-function $V(z) \in [E, E]$, ($\dim E < \infty$) of the class $N(R)$ will be said to be a member of the subclass $N_{01}(R)$ if the subspace

$$
E^1_{\infty} = \left\{ e \in E : \int_{-\infty}^{+\infty} (dG(t)e, e)_E < \infty \right\}
$$

possess a property: $E^1_{\infty} \not= \emptyset$, $E^1_{\infty} \not= E$.

**Theorem 2.** Let $\theta$ be a l.s.c.d. of the form (1), $\dim E < \infty$ where $A$ is a linear closed Hermitian operator with a dense domain and $\mathcal{D}(T) \not= \mathcal{D}(T^*)$. Then operator-function $V_0(z)$ of the form (5),(6) belongs to the class $N_0(R)$. Conversely, let an operator-function $V(z)$ act in a finitedimensional Hilbert space $E$ and belong to the class $N_0(R)$. Then it admits a minimal realization by the system $\theta$ of the form (1) with a preassigned direction operator $J(J = J^* = J^{-1} \in [E, E])$ and $A$ is a linear closed Hermitian operator with dense domain, $\mathcal{D}(T) \not= \mathcal{D}(T^*)$.

**Theorem 3.** Let $\theta$ be a l.s.c.d.s. of the form (1) where $\mathcal{D}(T) = \mathcal{D}(T^*)$ and $A$ is a linear closed regular Hermitian $O$-operator. Then an operator-function $V_0(z)$ of the form (5), (6) belongs to the class $N_1(R)$. Conversely, let an operator-function $V(z)$ act in a finitedimensional Hilbert space $E$ and belong to the class $N_1(R)$. Then it admits a minimal realization by the system $\theta$ of the form (1) with a preassigned direction operator $J(J = J^* = J^{-1} \in [E, E])$ where $A$ is a linear closed regular Hermitian $O$-operator with a non-dense domain and $\mathcal{D}(T) = \mathcal{D}(T^*)$.

The theorem, similar to theorems (1)-(3), takes place also for an operator-functions of the class $N_{01}(R)$. 5
DEFINITION 6. An operator-function \( W(z) \) acting in finitedimensional Hilbert space \( E \) is said to be a member of the class \( \Omega(R, J) \), \( \Omega_0(R, J), \Omega_1(R, J), \Omega_{01}(R, J) \), where \( J = J^* = J^{-1} \in [E, E] \), if it is holomorphic in some neighbourhood of the point \((-i)\) and an operator-function

\[
V(z) = i[W(z) + I]^{-1}[W(z) - I]J
\]

belongs to the class \( N(R) (N_0(R), N_1(R), N_{01}(R)) \) respectively.

Thus, classes \( \Omega(R, J), \Omega_0(R, J), \Omega_1(R, J), \Omega_{01}(R, J) \) represent the set of \( J \)-contractive operator-functions in the lower half-plane with properties (4), which can be realized as transfer operator-functions of the system \( \theta \) of the form (1) with those or other characteristics of the main operator \( T \).

The theorems (1)-(3) develop and make precise the known results by M.S.Livšic on the theory of inverse problem of the characteristic operator-functions and systems [10]. For the considered class of operators the theorem 3 reinforces one result from [9].

THEOREM 4. Let operator-functions \( W_1(z) \) and \( W_2(z) \) acting in finitedimensional Hilbert space \( E \) belong to classes \( \Omega(R, J) (\Omega_0(R, J), \Omega_1(R, J)) \). Then their product (in any order) also belongs to classes \( \Omega(R, J) (\Omega_0(R, J), \Omega_1(R, J)) \) respectively.

THEOREM 5. Let

\[
(7) \theta_1 = \begin{pmatrix} \kappa_1 & K_1 \\ \mathcal{H}_1 & J \end{pmatrix} \quad \text{and} \quad \theta_2 = \begin{pmatrix} \kappa_2 & K_2 \\ \mathcal{H}_2 & J \end{pmatrix}
\]

be linear stationary conservative dynamic systems of the form (1) so that their transfer operator-functions \( W_{\theta_1}(z) \) and \( W_{\theta_2}(z) \) belong to the class \( \Omega_{01}(R, J) \). The product \( W_{\theta_1}(z) \cdot W_{\theta_2}(z) \) \( (W_{\theta_2}(z) \cdot W_{\theta_1}(z)) \) also belongs to the class \( \Omega_{01}(R, J) \) if and only if the set

\[
\mathcal{D}_{12} = \{ x = x_1 + x_2 : x_1 \in \mathcal{D}(T_1^+), x_2 \in \mathcal{D}(T_2), K_1^* x_1 + K_2^* x_2 = 0 \}
\]

and respectively

\[
\mathcal{D}_{21} = \{ x = x_2 + x_1 : x_2 \in \mathcal{D}(T_2^+), x_1 \in \mathcal{D}(T_1, K_2^* x_2 + K_1^* x_1 = 0 \}
\]

is non-dense in Hilbert space \( \mathcal{H}_1 \oplus \mathcal{H}_2 \).

There exists an example of the systems \( \theta_1 \) and \( \theta_2 \) transfer mappings of which \( W_{\theta_1}(z) \) and \( W_{\theta_2}(z) \) belong to the class \( \Omega_{01}(R, J) \), but their product \( W_{\theta_1}(z) \cdot W_{\theta_2}(z) \) belongs to the class \( \Omega_0(R, J) \). The criterion when \( W_{\theta_1}(z) \cdot W_{\theta_2}(z) \) \( (W_{\theta_2}(z) \cdot W_{\theta_1}(z)) \) belongs to the class \( \Omega_{01}(R, J) \) if \( W_{\theta_1}(z) \) belongs to the \( \Omega_0(R, J) \) and \( W_{\theta_2}(z) \) belongs to the \( \Omega_1(R, J) \) is found. It may be shown also, that if \( W_1(z) \in \Omega_0(R, J), W_2(z) \in \Omega_1(R, J) \) and these operator-functions, acting in finitedimensional Hilbert space \( E \), are commuting, then \( W_1(z)W_2(z) \in \Omega_{01}(R, J) \).

Note, that theorem 4 permits system \( \theta \), the transfer mapping of which \( W_{\theta}(z) = W_{\theta_1}(z) \cdot W_{\theta_2}(z) \), to be built constructively, when transfer mappings \( W_{\theta_1}(z) \) and \( W_{\theta_2}(z) \) of systems \( \theta_1 \) and \( \theta_2 \) of the form (1) are known.
There is a procedure of "projection" of the system $\theta$ of the form (1) onto an arbitrary invariant subspace of the operator $T$ and its orthogonal complement. In addition, for systems $\theta_1$ and $\theta_2$, which are "projections" of the system $\theta$ a factorization formula

$$W_{\theta}(z) = W_{\theta_1}(z) \cdot W_{\theta_2}(z)$$

is valid. Besides, if a transfer mapping $W_{\theta}(z)$ belongs to the class $\Omega(R, J)$, then $W_{\theta_1}(z)$ and $W_{\theta_2}(z)$ also belong to the class $\Omega(R, J)$. The analogous property of factorization takes place for transfer mappings $W_{\theta}(z)$ of the class $\Omega_1(R, J)$. But if $W_{\theta}(z)$ belongs to the class $\Omega_0(R, J)$ or $\Omega_{01}(R, J)$, then in the factorization formula (8), as it follows from theorem 5 and comments to it, factors $W_{\theta_1}(z)$ and $W_{\theta_2}(z)$ can, generally speaking, be in different classes ($\Omega_0(R, J), \Omega_1(R, J), \Omega_{01}(R, J)$).

**Theorem 6.** Let $\theta$ be a l.s.c.d.s. of the form (1) with an invertible channel operator $K$ and a direction operator $J$ ($J = J^* = J^{-1} \in [E, E], \dim E < \infty$), where transfer mapping $W_{\theta}(z)$ belongs to the class $\Omega_0(R, J)$. Then for arbitrary constant $J$-unitary operators $B$ and $C$, acting in Hilbert space $E$, the product $B \cdot W_{\theta}(z) \cdot C$, also belongs to the class $\Omega_0(R, J)$.

Theorem 6 in somewhat other wording was established by Yu.M.Arlinskiĭ and E.R.Tsekanovskiĭ [1] and being published for the first time. Note that theorem 6 fails to be true if $\dim E = \infty$. See [1]. There is a procedure of realization $W(z) \cdot C$ ($B \cdot W(z)$) knowing realization $W(z) \in \Omega_0(R, J)$, where $B$ and $C$ are an arbitrary constant $J$-unitary operator [1]. Besides, it was established [1] that if $W_{\theta_1}(z)$ and $W_{\theta_2}(z)$ belong to the class $\Omega_0(R, J)$ and operators $A_1$ and $A_2$ of systems $\theta_1$ and $\theta_2$ of the form (7) are different correct ($*$)-extensions of the same operator $T$ of the class $\Lambda_A$, then under some restrictions of the channel operators $K_1$ and $K_2$ of systems $\theta_1$ and $\theta_2$, respectively, transfer mappings $W_{\theta_1}$ and $W_{\theta_2}$ satisfy the relation

$$W_{\theta_2}(z) = W_{\theta_1}(z) \cdot C$$

where $C$ is some constant $J$-unitary operator.

Note that theorems 1 - 6 are a further development and complement of the investigations by M.S.Brodskiĭ, M.S.Livšic, V.P.Potapov, A.V.Shtraus, N.Bart, I.Gohberg, M.Kaashoek, A.C.Ran [3] [4] [6] [10] [12] [13,14] (see, also survey [5]). Realization problems for a very general class of transfer functions of systems with, generally speaking, unbounded operators have recently been investigated by G.Weiss [15].

**References**


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