On the realization of inverse Stieltjes functions

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Abstract

In this paper realization problems for a class of operator-valued inverse Stieltjes functions acting on a finite-dimensional Hilbert space are considered. They appear as linear fractional transformations of the operator-valued transfer functions (characteristic functions) of linear stationary conservative dynamical systems (Brodskiĭ-Livšic rigged operator colligations). Proofs of both the direct and inverse realization theorems are provided.

1 Introduction

The major part of realization (representation) theory concerns the identification of a given holomorphic function as a transfer (characteristic) function of a system (colligation) or a linear fractional transformation of such a function. Systems whose main operator is bounded have been investigated thoroughly, and original results go back to the works of M.S. Brodskiĭ and M.S. Livšic, cf. [7], [12]. However many realizations in different fields including system theory, scattering theory, and electrical engineering involve unbounded main operators and a complete theory is not yet available. The aim of the present paper is to outline the necessary steps needed to obtain a more general realization theory for a special class of holomorphic functions along the lines of M.S. Brodskiĭ and M.S. Livšic.

An operator-valued function $V(z)$ acting on a Hilbert space $\mathfrak{E}$ belongs to the class $\mathbf{N}$ of Herglotz-Nevanlinna functions if outside $\mathbb{R}$ it is holomorphic, symmetric, i.e., $V^*(z) = V(\bar{z})$, and satisfies $(\operatorname{Im} z)(\operatorname{Im} V(z)) \geq 0$. Each Herglotz-Nevanlinna function $V(z)$ has an integral representation of the form

$$V(z) = Q + Fz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) dG(t), \tag{1.1}$$

where $Q = Q^*$, $F = F^* \geq 0$, and $G(t)$ is nondecreasing operator-valued function on $\mathbb{R}$, such that

$$\int_{\mathbb{R}} \frac{dG(t)}{t^2+1} < \infty,$$

cf. [4], [11], [14]. The class of Herglotz-Nevanlinna functions has important subclasses such as the class of Stieltjes functions and the class of inverse Stieltjes functions [11].

The question alluded to above is when a Herglotz-Nevanlinna function $V(z)$ can be considered to be the characteristic function of a system (colligation) or its linear fractional transformation. By a system is usually meant a linear stationary conservative dynamical system (l.s.c.d.s.) Θ of the form

$$\begin{cases} (\mathbb{A} - zI) = KJ\varphi_-, \\ \varphi_+ = \varphi_- - 2iK^*x, \end{cases} \qquad \operatorname{Im} \mathbb{A} = KJK^*, \tag{1.2}$$

where K and J are bounded linear operators in Hilbert spaces, φ_- is an input vector, φ_+ is an output vector, and x is an inner state vector of the system Θ. The main operator $\mathbb{A}$ of the system need not necessarily be bounded.

The realization of general Herglotz-Nevanlinna functions requires more general systems than (1.2); the theory remains to be worked out in detail, cf. [8], [9], [10]. However, various subclasses of Herglotz-Nevanlinna functions related to systems of the type (1.2) have been identified, see [1], [2], [3], [5]. In particular, in [16], [18] necessary and sufficient conditions were given for the main operator $\mathbb{A}$, so that the linear fractional transformation of its transfer function belongs to the class of Stieltjes functions. In the present paper similar questions are treated related to the class of inverse Stieltjes functions. The approach in this note is based on the use of rigged Hilbert spaces. This method for solving inverse problems in the theory of operator-valued characteristic functions was introduced in [17] and was further developed in [1], [4], [5].

2 Preliminaries

This section contains some basic definitions and results that will be used in the proof of the realization theorem for inverse Stieltjes functions. Let $\mathfrak{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$. Let A be a closed linear Hermitian operator in $\mathfrak{H}$, i.e. $(Af,g)=(f,Ag)$ for all $f,g\in\operatorname{dom}A$, whose domain $\operatorname{dom}A$ need not be dense. Let $\mathfrak{H}_0=\overline{\operatorname{dom}}\,A$ and let A^* be the adjoint of the operator A (as an operator from $\mathfrak{H}_0$ into $\mathfrak{H}$).

Since A is an operator it is clear that $\overline{\operatorname{dom}}\,A^*=\mathfrak{H}$. Now equip $\mathfrak{H}_+=\operatorname{dom}A^*$ with the inner product

$$(f,g)_+=(f,g)+(A^*f,A^*g),\quad f,g\in\mathfrak{H}_+, \tag{2.1}$$

and then construct a *rigged Hilbert space* triplet $\mathfrak{H}_+\subset\mathfrak{H}\subset\mathfrak{H}_-$ with positive and negative norms, respectively; cf. [6]. In the following, the symbols $(+)$ and $(-)$ indicate the norms $\|\cdot\|_+$ and $\|\cdot\|_-$ by which the geometrical and topological concepts are defined in $\mathfrak{H}_+$ and $\mathfrak{H}_-$. Denote

$$\mathfrak{M}_\lambda=\operatorname{ran}(A-\lambda I),\quad \mathfrak{N}_\lambda=\mathfrak{M}_{\bar\lambda}^{\perp}. \tag{2.2}$$

Here the subspace $\mathfrak{N}_\lambda$ is the *defect subspace* of A at $\bar\lambda$ and the numbers $\dim\mathfrak{N}_\lambda$ and $\dim\mathfrak{N}_{\bar\lambda}$, when $\operatorname{Im}\lambda<0$, are the *deficiency indices* of the operator A.

The set of all bounded linear operators acting from the Hilbert space $\mathfrak{H}_1$ into the Hilbert space $\mathfrak{H}_2$ is denoted by $[\mathfrak{H}_1,\mathfrak{H}_2]$. An operator $\mathbb{A}\in[\mathfrak{H}_+,\mathfrak{H}_-]$ is said to be a *bi-extension* of A if the inclusions $\mathbb{A}\supset A$ and $\mathbb{A}^*\supset A$ are both satisfied. If, in addition, $\mathbb{A}=\mathbb{A}^*$ then $\mathbb{A}$ is called a *self-adjoint bi-extension* of the operator A.

Given a closed Hermitian operator A, a closed densely defined linear operator T acting on the Hilbert space $\mathfrak{H}$ is said to *belong to the class* Ω_A if $T\supset A$, $T^*\supset A$, and $-i$ is a regular point of T.

An operator $\mathbb{A}$ in $[\mathfrak{H}_+,\mathfrak{H}_-]$ is called a *$(*)$-extension* of an operator T from the class Ω_A, if the inclusions $\mathbb{A}\supset T$ and $\mathbb{A}^*\supset T^*$ are both satisfied. A $(*)$-extension $\mathbb{A}$ in $[\mathfrak{H}_+,\mathfrak{H}_-]$ is said to be *correct* if $\mathbb{A}_R=\frac{1}{2}(\mathbb{A}+\mathbb{A}^*)$ satisfies

$$\mathbb{A}_R\supset\hat A=\hat A^*\supset A,$$

where $\hat A=\{\,\{f,\mathbb{A}_Rf\}:\,f\in\mathfrak{H},\ \mathbb{A}_Rf\in\mathfrak{H}\,\}$.

An operator $T\in\Omega_A$ is said to belong to *the class* Λ_A, if T admits correct $(*)$-extensions and A is the maximal common Hermitian part of T and T^*. It is known that for a closed Hermitian operator A with finite, equal defect indices, the classes Ω_A and Λ_A coincide.

Finally, recall that a closed Hermitian operator A is called *simple* if there is no non-trivial reducing subspace where it generates a self-adjoint operator. It is known, cf. e.g. [15], that a symmetric operator A with equal deficiency indices is simple if and only if

$$\overline{\operatorname{span}}\,\{\,\mathfrak{N}_\lambda:\ \lambda\neq\bar\lambda\,\}=\mathfrak{H}. \tag{2.3}$$

3 Linear stationary conservative dynamical systems

In this section linear stationary conservative dynamical systems Θ of the form (1.2) are generalized to allow triplets of Hilbert spaces.

Definition 3.1. *The array*

$$\Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & \mathfrak{E} \end{pmatrix} \tag{3.1}$$

is called a linear stationary conservative dynamical system (l. s. c. d. s.) or Brodskiĭ-Livšic rigged operator colligation if

(i) $\mathbb{A}$ *is a correct* $(*)$*-extension of an operator* T *of the class* Λ_A*;*

(ii) $J = J^* = J^{-1} \in [\mathfrak{E}, \mathfrak{E}]$, $\dim \mathfrak{E} < \infty$*;*

(iii) $\mathbb{A} - \mathbb{A}^* = 2iKJK^*$*, where* $K \in [E, \mathfrak{H}_-]$, $K^* \in [\mathfrak{H}_+, E]$.

In this case, the operator $\mathbb{A}$ ($\mathbb{A} \supset T \supset A$, $\mathbb{A}^* \supset T^* \supset A$) is called a *main operator*, the operator K is called a *channel operator*, and J is called a *direction operator*. A system Θ of the form (3.1) will be called a *scattering* system (*dissipative* operator colligation) if $J = I$. A system Θ is called *minimal* if its Hermitian operator A is simple.

The operator-valued function $W_\Theta(z)$ defined by

$$W_\Theta(z) = I - 2iK^*(\mathbb{A} - zI)^{-1}KJ, \tag{3.2}$$

is called an *operator-valued transfer function* of the system Θ or an operator-valued characteristic function of Brodskiĭ-Livšic rigged operator colligation. According to [13], $\operatorname{ran} K \subset \operatorname{ran}(\mathbb{A} - \lambda I)$, and therefore $W_\Theta(z)$ is well defined. Another operator-valued function related to the l.s.c.d.s. Θ in (3.1) is given by

$$V_\Theta(z) = K^*(\mathbb{A}_R - zI)^{-1}K. \tag{3.3}$$

The transfer function $W_\Theta(z)$ of the system Θ and the function $V_\Theta(z)$ of the form (3.3) are connected via

$$V_\Theta(z) = i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I]J. \tag{3.4}$$

Definition 3.2. *An operator-valued function* $V(z)$ *acting on a finite-dimensional Hilbert space* $\mathfrak{E}$ *is realizable, if in some neighborhood of the point* $-i$*, the function* $V(z)$ *can be represented in the form*

$$V(z) = i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I]J, \tag{3.5}$$

where $W_\Theta(z)$ *is the transfer function of some l.s.c.d.s.* Θ *with the direction operator* J ($J = J^* = J^{-1} \in [\mathfrak{E}, \mathfrak{E}]$).

It was established in [4] that an operator-valued Herglotz-Nevanlinna function $V(z) \in [\mathfrak{E}, \mathfrak{E}]$, $\dim \mathfrak{E} < \infty$, can be realized by the l.s.c.d.s. Θ of the type (3.3) if and only if in the representation (1.1) the following conditions are satisfied:

(i) $F = 0$;

(ii) $Qe = \int_{\mathbb{R}} \frac{t}{1+t^2} dG(t)e$ for all $e \in \mathfrak{E}$ such that $\int_{\mathbb{R}} (dG(t)e, e)_{\mathfrak{E}} < \infty$.

4 Realization theorems for the class $\mathbf{S}_0^{-1}$

The scalar version of the following definition can be found in [11]. In the present definitions it is assumed that the underlying space $\mathfrak{E}$ is finite-dimensional.

Definition 4.1. *An operator-valued Herglotz-Nevanlinna function $V(z) \in [\mathfrak{E}, \mathfrak{E}]$ is called a Stieltjes or $\mathbf{S}$-function if $V(z)$ is holomorphic in $\mathrm{Ext}[0, +\infty)$ and $V(z) \geq 0$ in $(-\infty, 0)$, and it is called an inverse Stieltjes function or $\mathbf{S}^{-1}$-function if $V(z)$ is holomorphic in $\mathrm{Ext}[0, +\infty)$ and $V(z) \leq 0$ in $(-\infty, 0)$.*

The following two lemmas can be easily obtained from their scalar versions in [11].

Lemma 4.1. *An invertible operator-valued function $V(z) \in [\mathfrak{E}, \mathfrak{E}]$ ($V(z) \neq 0$) is an $\mathbf{S}^{-1}$-function if and only if the function $-V(z)^{-1}$ is an $\mathbf{S}$-function.*

Lemma 4.2. *An operator-valued function $V(z) \in [\mathfrak{E}, \mathfrak{E}]$ is an $\mathbf{S}^{-1}$-function if and only if $V(z)$ and $V(z)/z$ both are operator-valued Herglotz-Nevanlinna functions.*

Similar to (1.1) there are integral representations for Stieltjes and inverse Stieltjes functions. In particular, $V(z)$ is an operator-valued inverse Stieltjes function if and only if

$$V(z) = \alpha + \beta z + \int_{0+}^{\infty} \left(\frac{1}{t-z} - \frac{1}{t} \right) dG(t), \tag{4.1}$$

where $\alpha \leq 0$, $\beta \geq 0$, and $G(t)$ is a non-decreasing operator-valued function on $[0, +\infty)$, such that

$$\int_{0+}^{\infty} \frac{dG(t)}{t+t^2} \in [\mathfrak{E}, \mathfrak{E}].$$

The next definition is motivated by the general realization result from [4], mentioned directly after Definition 3.2.

Definition 4.2. *An operator-valued $\mathbf{S}^{-1}$-function $V(z) \in [\mathfrak{E}, \mathfrak{E}]$ belongs to the class $\mathbf{S}_0^{-1}$ if in the representation (4.1) the following conditions are satisfied:*

(i) $\beta = 0$;

(ii) $\int_{0+}^{\infty} (dG(t)e, e)_E = \infty$ *for all* $e \in \mathfrak{E}$.

The next two theorems are the main results in this note. The first of them gives some sufficient conditions for the system Θ of the form (1.2) which guarantee that $V_\Theta(z)$ is an inverse Stieltjes function from the class $\mathbf{S}_0^{-1}$.

Theorem 4.1. *Let Θ be an l.s.c.d.s of the form (3.1) with $\overline{\operatorname{dom} A} = \mathfrak{H}$, and assume that*

$$(\mathbb{A}_R f, f) \leq (A^* f, f) + (f, A^* f), \quad f \in \mathfrak{H}_+. \tag{4.2}$$

Then the function $V_\Theta(z)$ of the form (3.3), (3.4) belongs to the class $\mathbf{S}_0^{-1}$.

Proof. First it will be shown that $V_\Theta(z)$ belongs to $\mathbf{S}^{-1}$. Let $\{z_k\}$, $k = 1, ..., n$, be a sequence of non-real complex numbers and let φ_k be a sequence of elements of $\mathfrak{N}_{z_k}$, the defect subspace of A. Then for every φ_k there exists $h_k \in \mathfrak{E}$ such that

$$\varphi_k = (A_R - z_k I)^{-1} K h_k, \quad k = 1, \dots, n, \tag{4.3}$$

and conversely, see [17]. With $\varphi = \sum_{k=1}^n \varphi_k / z_k$ it follows from $A^* \varphi_k = z_k \varphi_k$ and (4.3) that

$$\begin{aligned}
&(A^*\varphi, \varphi) + (\varphi, A^*\varphi) - (\mathbb{A}_R \varphi, \varphi) \\
&= \sum_{k,l=1}^n \frac{1}{z_k \bar{z}_l} \left[(A^*\varphi_k, \varphi_l) + (\varphi_k, A^*\varphi_l) - (\mathbb{A}_R \varphi_k, \varphi_l)\right] \\
&= \sum_{k,l=1}^n \frac{1}{z_k \bar{z}_l} \left([-\mathbb{A}_R + z_k + \bar{z}_l]\varphi_k, \varphi_l\right) \\
&= \sum_{k,l=1}^n \left(\frac{(\mathbb{A}_R - \bar{z}_l I)^{-1}(\bar{z}_l(\mathbb{A}_R - \bar{z}_l I) - z_k(\mathbb{A}_R - z_k I))(\mathbb{A}_R - z_k I)^{-1}}{z_k \bar{z}_l (z_k - \bar{z}_l)} K h_k, K h_l \right) \\
&= \sum_{k,l=1}^n \left(\frac{\bar{z}_l K^*(\mathbb{A}_R - z_k I)^{-1} K - z_k K^*(\mathbb{A}_R - z_l I)^{-1} K}{z_k \bar{z}_l (z_k - \bar{z}_l)} h_k, h_l \right) \\
&= \sum_{k,l=1}^n \left(\frac{\bar{z}_l V_\Theta(z_k) - z_k V_\Theta(\bar{z}_l)}{z_k \bar{z}_l (z_k - \bar{z}_l)} h_k, h_l \right) \\
&= \sum_{k,l=1}^n \left(\frac{V_\Theta(z_k)/z_k - V_\Theta(\bar{z}_l)/\bar{z}_l}{z_k - \bar{z}_l} h_k, h_l \right).
\end{aligned}$$

In particular, one obtains

$$\left(\frac{V_\Theta(z)/z - V_\Theta(\bar{z})/\bar{z}}{z - \bar{z}} h, h \right) \geq 0, \quad h \in \mathfrak{E}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

so that $(\operatorname{Im} z)(\operatorname{Im}(V_\Theta(z)/z)) \geq 0$. Therefore, $V_\Theta(z)/z$ is an operator-valued Herglotz-Nevanlinna function. Moreover, $V_\Theta(z)$ itself is an operator-valued Herglotz-Nevanlinna function, cf. [4]. According to Lemma 4.2 this means that $V_\Theta(z)$ is an inverse Stieltjes function.

Next it will be shown that $V_\Theta(z)$ belongs to the class $\mathbf{S}_0^{-1}$. Since $V_\Theta(z)$ is an inverse Stieltjes function, (4.1) shows that

$$V_\Theta(z) = \alpha + \beta z + \int_{0+}^{\infty} \left(\frac{1}{t-z} - \frac{1}{t} \right) dG(t),$$

where $\alpha \leq 0$, $\beta \geq 0$, and

$$\int_{0+}^{\infty} \frac{dG(t)}{t+t^2} \in [\mathfrak{E}, \mathfrak{E}].$$

In a neighborhood of zero the function $t+t^2$ is equivalent to the function $t+t^3$ and in a neighborhood of ∞ one has the inequality

$$\frac{1}{t+t^3} < \frac{1}{t+t^2}.$$

Hence,

$$\int_{0+}^{\infty} \frac{dG(t)}{t+t^3} \in [\mathfrak{E}, \mathfrak{E}]. \tag{4.4}$$

Furthermore,

$$\begin{aligned} V_\Theta(z) &= \alpha + \beta z + \int_{0+}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} + \frac{t}{1+t^2} - \frac{1}{t} \right) dG(t) \\ &= \left(\alpha - \int_{0+}^{\infty} \frac{dG(t)}{t+t^3} \right) + \beta z + \int_{0+}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) dG(t). \end{aligned} \tag{4.5}$$

Assume that the function $G(t)$ has been properly normalized. Then the integral representation (4.5) coincides with the integral representation (1.1). By assumption the Herglotz-Nevanlinna function $V_\Theta(z)$ is realizable, so that $F = 0$, and consequently, $\beta = 0$. Hence, $V_\Theta(z)$ satisfies the first condition (i) in Definition 4.2. The second condition (ii) in Definition 4.2 is also satisfied, since $\overline{\operatorname{dom} A} = \mathfrak{H}$ (see [5], [13]). □

The next result is a converse to Theorem 4.1.

Theorem 4.2. *Let the operator-valued function $V(z)$ act in a finite-dimensional Hilbert space $\mathfrak{E}$, and assume that $V(z) \in \mathbf{S}_0^{-1}$. Then $V(z)$ admits a realization by a minimal system Θ of the form* (3.1) *with a nonnegative densely defined Hermitian operator A, and a preassigned direction operator J for which $I + iV(-i)J$ is invertible. Moreover, the condition* (4.2) *holds for this system.*

Proof. Each inverse Stieltjes function is a Herglotz-Nevanlinna function. By direct inspection it can be seen that an inverse Stieltjes function of the class $\mathbf{S}_0^{-1}$ can be realized by a minimal l.s.c.d.s., see the conditions stated in the end of Section 3 (cf. [4]). Thus it suffices to show that the realizing system in the proof of the realization theorem in [4] appears to be a system of the form (3.1) where the operator A is a densely defined nonnegative operator, such that condition (4.2) holds. The operator A in the proof of the realization theorem in [4] has the form

$$Af(t) = tf(t), \tag{4.6}$$

and acts in the model Hilbert space $L^2_G[0,+\infty)$. Hence, A is a nonnegative operator. The construction of $\operatorname{dom} A$ in [4] (see also [5]) immediately implies that $\overline{\operatorname{dom} A} = \mathfrak{H}$.

It remains to show that the system also satisfies the condition (4.2). Due to the construction of $\operatorname{dom} A$ in [4] and the fact that $\overline{\operatorname{dom} A} = \mathfrak{H}$, the operator A has equal deficiency indices. By construction, the system Θ is minimal, and therefore, the operator A is simple. Hence,

$$\overline{\operatorname{span}} \left\{ \mathfrak{N}_\lambda : \lambda \neq \bar{\lambda} \right\} = \mathfrak{H}_+, \tag{4.7}$$

where the closure is taken with respect to the $(+)$-metric. In the proof of Theorem 4.1 it has been shown that

$$(\mathbb{A}_R\varphi, \varphi) \leq (A^*\varphi, \varphi) + (\varphi, A^*\varphi), \quad \varphi = \sum_{k=1}^{n} \varphi_k / z_k, \quad \varphi_k \in \mathfrak{N}_{z_k}, \tag{4.8}$$

is equivalent to

$$\sum_{k,l=1}^{n} \left(\frac{V_\Theta(z_k)/z_k - V_\Theta(\bar{z}_l)/\bar{z}_l}{z_k - \bar{z}_l} h_k, h_l \right) \geq 0,$$

cf. (4.3). Now, combine (4.7) and (4.8) to obtain the inequality (4.2). Therefore, Θ is the required system. □

Observe, that for scattering systems, i.e. if $J = I$, the invertibility condition in Theorem 4.2 is automatically satisfied.

5 A class of Schrödinger operators

In this section a system Θ of the form (3.1) is constructed for non-selfadjoint Schrödinger operators on a half-line $[a,\infty)$ which are in the limit point case at ∞. Let $\mathfrak{H} = L^2[a,+\infty)$ and let $l(y) = -y'' + q(x)y$, where $q(x)$ is a real-valued locally summable function, such that the limit point case prevails at ∞. Then the symmetric operator

$$\begin{cases} Ay = -y'' + q(x)y, \\ y(a) = y'(a) = 0, \end{cases} \tag{5.1}$$

has deficiency indices $(1,1)$. Provide the linear space $\mathfrak{H}_+ = \operatorname{dom} A^*$ with the scalar product

$$(y,z)_+ = (y,z) + (l(y), l(z)), \quad y, z \in \operatorname{dom} A^*, \tag{5.2}$$

where $\operatorname{dom} A^*$ is the set of all absolutely continuous (including their derivatives) functions y for which $l(y) \in L^2[a,+\infty)$ and where $(\cdot,\cdot)$ stands for the scalar product in $L^2[a,+\infty)$. Construct a rigged Hilbert space

$$\mathfrak{H}_+ \subset L^2[a,+\infty) \subset \mathfrak{H}_-,$$

and, for $h \in \mathbb{C}$, $(\operatorname{Im} h > 0)$ consider the operators

$$\begin{cases} T_h y = -y'' + q(x)y, \\ hy(a) - y'(a) = 0, \end{cases} \tag{5.3}$$

and

$$\begin{cases} T_h^* y = -y'' + q(x)y, \\ \bar{h}y(a) - y'(a) = 0, \end{cases} \tag{5.4}$$

of the class Ω_A. It is known [16] that the operators

$$\begin{aligned} \mathbb{A}y &= -y'' + q(x)y + \frac{i}{\operatorname{Im} h}[hy(a) - y'(a)][\delta'(x-a) - m_\infty(0-)\delta(x-a)], \\ \mathbb{A}^*y &= -y'' + q(x)y - \frac{i}{\operatorname{Im} h}[\bar{h}y(a) - y'(a)][\delta'(x-a) - m_\infty(0-)\delta(x-a)], \end{aligned} \tag{5.5}$$

define correct $(*)$-extensions of the Schrödinger operators T_h and T_h^* of the form (5.3) and (5.4). Here $m_\infty(\lambda)$ is the Weyl function [16], while $\delta(x-a)$ and $\delta'(x-a)$ are the δ-function and its derivative, respectively. It is easy to check (see also [16]) that

$$\operatorname{Im} \mathbb{A} = \frac{1}{\operatorname{Im} h}[-y'(a) - m_\infty(0-)y(a)][\delta'(x-a) - m_\infty(0-)\delta(x-a)] = (\cdot, g)g, \tag{5.6}$$

where

$$g = \frac{1}{(\operatorname{Im} h)^{1/2}}\big[\delta'(x-a) - m_\infty(0-)\delta(x-a)\big]. \tag{5.7}$$

Let $\mathfrak{E} = \mathbb{C}^1$, and let K be defined by

$$K : c \mapsto cg, \quad c \in \mathbb{C}^1, \tag{5.8}$$

where $g \in \mathfrak{H}_+$ is given by (5.7). Clearly,

$$K^* y = (y, g), \quad y \in \mathfrak{H}_+,$$

and $\operatorname{Im} \mathbb{A} = K\,K^*$. The system

$$\Theta = \begin{pmatrix} \mathbb{A} & K & I \\ \mathfrak{H}_+ \subset L^2[a, +\infty) \subset \mathfrak{H}_- & & \mathbb{C}^1 \end{pmatrix} \tag{5.9}$$

is a l.s.c.d.s. with the main operator $\mathbb{A}$ of the form (5.5), the direction operator $J = I$, and the channel operator K defined by (5.8). It satisfies the conditions of the system in Theorem 4.2. Direct calculations show that

$$W_\Theta(\lambda) = -\frac{m_\infty(\lambda) + \bar{h}}{m_\infty(\lambda) + h}. \tag{5.10}$$

Consequently, the function

$$V_\Theta(\lambda) = i[W_\Theta(\lambda) + I]^{-1}[W_\Theta(\lambda) - I]$$

is an inverse Stieltjes function of the class $\mathbf{S}_0^{-1}$.

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