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SECTORIAL CLASSES OF INVERSE STIELTJES FUNCTIONS AND L-SYSTEMS

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Dedicated to the memory of Anatoly Gordeevich Kostuchenko

Abstract. We introduce sectorial classes of inverse Stieltjes functions acting on a finite-dimensional Hilbert space as well as scalar classes of inverse Stieltjes functions based upon their limit behavior at minus infinity and at zero. It is shown that a function from these classes can be realized as the impedance function of a singular L-system and the operator $\tilde{\mathbb{A}}$ in a rigged Hilbert space associated with the realizing system is sectorial. Moreover, it is established that the knowledge of the limit values of the scalar impedance function allows to find an angle of sectoriality of the operator $\tilde{\mathbb{A}}$ as well as the exact angle of sectoriality of the accretive main operator T of such a system. The corresponding new formulas connecting the limit values of the impedance function and the angle of sectoriality of $\tilde{\mathbb{A}}$ are provided. Application of these formulas yields that the exact angle of sectoriality of operators $\tilde{\mathbb{A}}$ and T is the same if and only if the limit value at zero of the corresponding impedance function (along the negative x-axis) is equal to zero. Examples of the realizing L-systems based upon the Schrödinger operator on half-line are presented.

1. Introduction

An operator-valued function $V(z)$ acting on a finite-dimensional Hilbert space E is called the Herglotz-Nevanlinna function if it is holomorphic on $\mathbb{C}\setminus\mathbb{R}$, symmetric with respect to the real axis, i.e., $V(z)^*=V(\bar z)$, $z\in\mathbb{C}\setminus\mathbb{R}$, and if it satisfies the positivity condition

$$\operatorname{Im} V(z)\geq 0,\quad z\in\mathbb{C}_+, \tag{1}$$

or equivalently if

$$\sum_{k,l=1}^{n}\left(\frac{V(z_k)-V(\bar z_l)}{z_k-\bar z_l}h_k,h_l\right)_E\geq 0 \tag{2}$$

holds for an arbitrary choice of non-real numbers $\{z_k\}$ and $\{h_k\}\in E$.

In the current paper we are going to focus on an important subclass of Herglotz-Nevanlinna functions, the inverse Stieltjes functions that can be realized as impedance functions of some singular L-systems. The formal definition, integral representation for inverse Stieltjes functions as well as the basic realization results are given in Sections 2 and 3.

In Section 4, which contains the main results of the present paper, we introduce the so called *sectorial* class $S^{-1,\alpha}$ of inverse Stieltjes functions in Hilbert space E and a class S^{-1,α_1,α_2} of scalar inverse Stieltjes functions based upon their limits (along the negative x-axis) at minus infinity and at zero. The theorems for these sectorial classes presented in Section 4 allow us to observe the geometric properties of the realizing L-systems whose

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impedance functions belong to $S^{-1,\alpha}$. In particular, it is shown that impedance function $V(z)$ of L-system acting on finite-dimensional Hilbert space E belongs to $S^{-1,\alpha}$ if and only if an operator $\tilde{\mathbb{A}}$ (in triplets of rigged Hilbert spaces) associated with L-system is α-sectorial. The relationship between the scalar classes $S^{-1,\alpha}$ and S^{-1,α_1,α_2} are established. In particular, using the function limit values at zero and infinity permits us to find an angle of sectoriality of the operator $\tilde{\mathbb{A}}$ as well as the exact angle of sectoriality of the main operator T of L-system. This approach also allows us to discover that the exact angles of sectoriality of operators T and $\tilde{\mathbb{A}}$ are equal if and only if the limit value at zero along negative x-axis of impedance function $V(z)$ is equal to zero. Section 5 concludes our paper by providing important illustrations of the results of Section 4 as applied to L-systems with a Schrödinger operator.

2. Preliminaries

For a pair of Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ we denote by $[\mathcal{H}_1, \mathcal{H}_2]$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Let $\dot{A}$ be a closed, densely defined, symmetric operator in a Hilbert space $\mathcal{H}$ with inner product $(f,g), f,g \in \mathcal{H}$. Any operator T in $\mathcal{H}$ such that

$$\dot{A} \subset T \subset \dot{A}^*$$

is called a *quasi-self-adjoint extension* of $\dot{A}$.

Consider the rigged Hilbert space (see [4], [2]) $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$, where $\mathcal{H}_+ = \mathrm{Dom}(\dot{A}^*)$ and

$$(f,g)_+ = (f,g) + (\dot{A}^* f, \dot{A}^* g), \quad f,g \in \mathrm{Dom}(A^*). \tag{3}$$

Let $\mathcal{R}$ be the *Riesz-Berezansky operator* $\mathcal{R}$ (see [4], [2]) which maps $\mathcal{H}_-$ onto $\mathcal{H}_+$ such that $(f,g) = (f, \mathcal{R}g)_+$ ($\forall f \in \mathcal{H}_+,\ g \in \mathcal{H}_-$) and $\|\mathcal{R}g\|_+ = \|g\|_-$. Note that identifying the space conjugate to $\mathcal{H}_\pm$ with $\mathcal{H}_\mp$, we get that if $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$, then $\mathbb{A}^* \in [\mathcal{H}_+, \mathcal{H}_-]$.

Definition 1. *An operator $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a self-adjoint bi-extension of a symmetric operator $\dot{A}$ if $\mathbb{A} = \mathbb{A}^*$ and $\mathbb{A} \supset \dot{A}$.*

Let $\mathbb{A}$ be a self-adjoint bi-extension of $\dot{A}$ and let the operator $\widehat{A}$ in $\mathcal{H}$ be defined as follows:

$$\mathrm{Dom}(\widehat{A}) = \{f \in \mathcal{H}_+ : \widehat{A} f \in \mathcal{H}\}, \quad \widehat{A} = \mathbb{A}\upharpoonright \mathrm{Dom}(\widehat{A}).$$

The operator $\widehat{A}$ is called a *quasi-kernel* of a self-adjoint bi-extension $\mathbb{A}$ (see [8], [1]).

Definition 2. *Let T be a quasi-self-adjoint extension of $\dot{A}$ with nonempty resolvent set $\rho(T)$. An operator $\mathbb{A} \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a $(*)$-extension of an operator T if*

1. $\mathbb{A} \supset T \supset \dot{A}, \quad \mathbb{A}^* \supset T^* \supset \dot{A}$,
2. *the quasi-kernel of self-adjoint bi-extension* $\mathrm{Re}\,\mathbb{A} = \frac{1}{2}(\mathbb{A} + \mathbb{A}^*)$ *is a self-adjoint extension of $\dot{A}$.*

In what follows we assume that $\dot{A}$ has equal finite deficiency indices and will say that a quasi-self-adjoint extension T of $\dot{A}$ belongs to the *class* $\Lambda(\dot{A})$ if $\rho(T) \neq \emptyset$, $\mathrm{Dom}(\dot{A}) = \mathrm{Dom}(T) \cap \mathrm{Dom}(T^*)$, and hence T admits $(*)$-extensions. The description of all $(*)$-extensions via Riesz-Berezansky operator $\mathcal{R}$ can be found in [1].

Recall that a linear operator T in a Hilbert space $\mathfrak{H}$ is called **accretive** [6] if $\mathrm{Re}\,(Tf,f) \geq 0$ for all $f \in \mathrm{Dom}(T)$. We call an accretive operator T **α-sectorial** [6] if there exists a value of $\alpha \in (0, \pi/2)$ such that

$$|\,\mathrm{Im}(Tf,f)| \leq (\tan\alpha)\,\mathrm{Re}\,(Tf,f), \quad f \in \mathrm{Dom}(T). \tag{4}$$

We say that the angle of sectoriality α is **exact** for an α-sectorial operator T if

$$\tan\alpha = \sup_{f\in \mathrm{Dom}(T)} \frac{|\,\mathrm{Im}(Tf,f)|}{\mathrm{Re}\,(Tf,f)}.$$

A bi-extension $\mathbb{A}$ of $\dot{A}$ is called **accretive** if $\mathrm{Re}\,(\mathbb{A}f,f)\geq 0$ for all $f\in\mathcal{H}_+$. This is equivalent to that the real part $\mathrm{Re}\,\mathbb{A}=(\mathbb{A}+\mathbb{A}^*)/2$ is a nonnegative self-adjoint bi-extension of $\dot{A}$.

Definition 3. *A system of equations*

$$\begin{cases} (\mathbb{A}-zI)x = KJ\varphi_-,\\ \varphi_+ = \varphi_- - 2iK^*x, \end{cases}$$

or an array

$$\Theta = \begin{pmatrix} \mathbb{A} & K & J\\ \mathcal{H}_+\subset\mathcal{H}\subset\mathcal{H}_- & & E \end{pmatrix} \tag{5}$$

is called an ***L-system*** *if*

(1) $\mathbb{A}$ *is a* $(*)$*-extension of an operator* T *of the class* $\Lambda(\dot{A})$;
(2) $J=J^*=J^{-1}\in[E,E]$, $\dim E<\infty$;
(3) $\mathrm{Im}\,\mathbb{A}=KJK^*$, *where* $K\in[E,\mathcal{H}_-]$, $K^*\in[\mathcal{H}_+,E]$, *and* $\mathrm{Ran}(K)=\mathrm{Ran}(\mathrm{Im}\,\mathbb{A})$.

In the definition above $\varphi_-\in E$ stands for an input vector, $\varphi_+\in E$ is an output vector, and x is a state space vector in $\mathcal{H}$. An operator $\mathbb{A}$ is called a *state-space operator* of the system Θ, J is a *direction operator*, and K is a *channel operator*. A system Θ in (5) is called *minimal* if the operator $\dot{A}$ is a prime operator in $\mathcal{H}$, i.e., there exists no non-trivial reducing invariant subspace of $\mathcal{H}$ on which it induces a self-adjoint operator. A system Θ in (5) is called *scattering* if $J=I$.

We associate with an L-system Θ the operator-valued function

$$W_\Theta(z)=I-2iK^*(\mathbb{A}-zI)^{-1}KJ,\quad z\in\rho(T), \tag{6}$$

which is called the **transfer function** of the L-system Θ. We also consider the operator-valued function

$$V_\Theta(z)=K^*(\mathrm{Re}\,\mathbb{A}-zI)^{-1}K. \tag{7}$$

It was shown in [2], [1] that both (6) and (7) are well defined. The transfer operator-function $W_\Theta(z)$ of the system Θ and an operator-function $V_\Theta(z)$ of the form (7) are connected by the following relations valid for $\mathrm{Im}\,z\neq 0$, $z\in\rho(T)$,

$$\begin{aligned} V_\Theta(z)&=i[W_\Theta(z)+I]^{-1}[W_\Theta(z)-I]J,\\ W_\Theta(z)&=(I+iV_\Theta(z)J)^{-1}(I-iV_\Theta(z)J). \end{aligned}$$

The function $V_\Theta(z)$ defined by (7) is called the **impedance function** of an L-system Θ of the form (5). The class of all Herglotz-Nevanlinna functions in a finite-dimensional Hilbert space E, that can be realized as impedance functions of an L-system, was described in [2].

3. Realization of inverse Stieltjes functions

Let E be a finite-dimensional Hilbert space. A scalar version of the following definition can be found in [5].

Definition 4. *We will call an operator-valued Herglotz-Nevanlinna function $V(z)$ in a finite-dimensional Hilbert space E by an* ***inverse Stieltjes*** *if $V(z)$ it is holomorphic in* $\mathrm{Ext}[0,+\infty)$ *and*

$$\frac{\mathrm{Im}[V(z)/z]}{\mathrm{Im}\,z}\geq 0. \tag{8}$$

Combining (8) with (2) we obtain

$$\sum_{k,l=1}^{n}\left(\frac{V(z_k)/z_k - V(\bar z_l)/\bar z_l}{z_k-\bar z_l}h_k, h_l\right)_E \geq 0$$

for an arbitrary sequence $\{z_k\}$ $(k=1,\dots,n)$ of complex numbers and a sequence of vectors $\{h_k\}$ in E. It can be shown (see [5]) that every inverse Stieltjes function $V(z)$ in a finite-dimensional Hilbert space E admits the following integral representation:

$$V(z)=\gamma+z\beta+\int_0^\infty\left(\frac{1}{t-z}-\frac{1}{t}\right)dG(t), \tag{9}$$

where $\gamma\leq 0$, $\beta\geq 0$, and $G(t)$ is a non-decreasing on $[0,+\infty)$ operator-valued function such that

$$\int_0^\infty\frac{(dG(t)h,h)}{t+t^2}<\infty,\quad \forall h\in E.$$

The following definition provides the description of a realizable subclass of inverse Stieltjes operator-valued functions.

Definition 5. *An operator-valued inverse Stieltjes function $V(z)$ in a finite-dimensional Hilbert space E is a member of the* ***class*** $S_0^{-1}(R)$ *if in the representation (9) we have*

- $\beta=0$,
- $\int_0^\infty (dG(t)h,h)_E=\infty$ *for all* $h\in E$, $h\neq 0$.

A $(*)$-extensions $\mathbb{A}$ of an operator $T\in\Lambda(\dot A)$ is called **accumulative** if

$$(\operatorname{Re}\mathbb{A}f,f)\leq(\dot A^*f,f)+(f,\dot A^*f),\quad f\in\mathcal{H}_+. \tag{10}$$

An L-system Θ of the form (5) is called **accumulative** if its operator $\mathbb{A}$ is accumulative, i.e., satisfies (10). It is easy to see that if an L-system is accumulative, then (10) implies that the operator $\dot A$ of the system is non-negative and both operators T and T^* are accretive. We also associate another operator $\tilde{\mathbb{A}}$ to an accumulative L-system Θ. It is given by

$$\tilde{\mathbb{A}}=2\operatorname{Re}\dot A^*-\mathbb{A}, \tag{11}$$

where $\dot A^*$ is in $[\mathcal{H}_+,\mathcal{H}_-]$. Obviously, $\operatorname{Re}\dot A^*\in[\mathcal{H}_+,\mathcal{H}_-]$ and $\tilde{\mathbb{A}}\in[\mathcal{H}_+,\mathcal{H}_-]$. Clearly, $\tilde{\mathbb{A}}$ is a bi-extension of $\dot A$ and is accretive if and only if $\mathbb{A}$ is accumulative. It is also not hard to see that even though $\tilde{\mathbb{A}}$ is not a $(*)$-extensions of the operator T but the form $(\tilde{\mathbb{A}}f,f)$, $f\in\mathcal{H}_+$ extends the form (f,Tf), $f\in\operatorname{Dom}(T)$.

The following statement [1] is the direct realization theorem for the functions of the class $S_0^{-1}(R)$.

Theorem 6. *Let Θ be an accumulative L-system of the form* (5) *with an invertible channel operator K and $\overline{\operatorname{Dom}(\dot A)}=\mathcal{H}$. Then its impedance function $V_\Theta(z)$ of the form* (7) *belongs to the class $S_0^{-1}(R)$.*

The inverse realization theorem [1] can be stated for the class $S_0^{-1}(R)$ as follows.

Theorem 7. *Let an operator-valued function $V(z)$ belong to the class $S_0^{-1}(R)$. Then $V(z)$ can be realized as an impedance function of an accumulative minimal L-system Θ of the form* (5) *with an invertible channel operator K, a non-negative densely defined symmetric operator $\dot A$ and $J=I$.*

4. Sectorial classes $S^{-1,\alpha}$ and S^{-1,α_1,α_2} and their realizations

Let $\alpha \in (0, \frac{\pi}{2})$. We introduce **sectorial subclasses** $S^{-1,\alpha}$ of operator-valued inverse Stieltjes functions as follows. An operator-valued inverse Stieltjes function $V(z)$ belongs to $S^{-1,\alpha}$ if

$$K_\alpha = \sum_{k,l=1}^{n} \left(\left[\frac{V(z_k)/z_k - V(\bar{z}_l)/\bar{z}_l}{z_k - \bar{z}_l} - (\cot\alpha)\, \frac{V^*(z_l)}{\bar{z}_l} \frac{V(z_k)}{z_k} \right] h_k, h_l \right)_E \geq 0 \tag{12}$$

for an arbitrary sequence $\{z_k\}$ $(k = 1, \dots, n)$ of complex numbers and a sequence of vectors $\{h_k\}$ in E. For $0 < \alpha_1 < \alpha_2 < \frac{\pi}{2}$, we have

$$S^{-1,\alpha_1} \subset S^{-1,\alpha_2} \subset S^{-1},$$

where S^{-1} denotes the class of all inverse Stieltjes functions (which corresponds to the case $\alpha = \frac{\pi}{2}$), as follows from the inequality

$$K_{\alpha_1} \leq K_{\alpha_2} \leq K_{\frac{\pi}{2}}.$$

The following theorem refines the result of Theorems 6 and 7 as applied to the class $S^{-1,\alpha}$.

Theorem 8. *Let Θ be an accumulative scattering minimal L-system of the form* (5) *with an invertible channel operator K and $\overline{\mathrm{Dom}(\dot{A})} = \mathcal{H}$. Then the impedance function $V_\Theta(z)$ defined by* (7) *belongs to the class $S^{-1,\alpha}$ if and only if the operator $\tilde{\mathbb{A}}$ of the form* (11) *associated to the L-system Θ is α-sectorial.*

Proof. Relying on the results of Theorems 6 and 7 all we need is to show that (12) is equivalent to relation (4) in the definition of sectoriality. Suppose that $\tilde{\mathbb{A}}$ is α-sectorial, then (4) holds for all $g \in \mathcal{H}_+$ and hence

$$\cot\alpha \cdot |(\mathrm{Im}\,\tilde{\mathbb{A}}\, g, g)| \leq (\mathrm{Re}\,\tilde{\mathbb{A}}\, g, g), \quad g \in \mathcal{H}_+. \tag{13}$$

Let $\{z_k\}$ $(k = 1, \dots, n)$ be a sequence of $(\mathrm{Im}\, z_k > 0)$ complex numbers and h_k be a sequence of vectors in E. Let us denote

$$K h_k = \delta_k, \quad g_k = (\mathrm{Re}\,\mathbb{A} - z_k I)^{-1} \delta_k, \quad g = \sum_{k=1}^{n} g_k. \tag{14}$$

Consequently, it follows from (4) and (14) that

$$\sum_{k,\ell=1}^{n} (\mathrm{Re}\,\tilde{\mathbb{A}}\, g_k, g_\ell) \geq (\cot\alpha) \Big| \sum_{k,\ell=1}^{n} (\mathrm{Im}\,\tilde{\mathbb{A}}\, g_k, g_\ell) \Big|. \tag{15}$$

Let φ_k be a sequence of elements of $\mathfrak{N}_{z_k}$ $(z_k \neq \bar{z}_k)$, the defect subspace of the operator $\dot{A}$. Then for every k there exists $h_k \in E$ such that

$$\varphi_k = \frac{1}{z_k} (\mathrm{Re}\,\mathbb{A} - z_k I)^{-1} K h_k \quad (k = 1, \dots, n). \tag{16}$$

Taking into account that $\dot{A}^*\varphi_k = z_k\varphi_k$, formulas (11) and (16), and letting $\varphi = \sum_{k=1}^n \varphi_k$ we get

$$\begin{aligned}
(\operatorname{Re}\tilde{\mathbb{A}}\varphi,\varphi) &= (\dot{A}^*\varphi,\varphi) + (\varphi,\dot{A}^*\varphi) - (\operatorname{Re}\mathbb{A}\varphi,\varphi)\\
&= \sum_{k,l=1}^n \Big[(\dot{A}^*\varphi_k,\varphi_l) + (\varphi_k,\dot{A}^*\varphi_l) - (\operatorname{Re}\mathbb{A}\varphi_k,\varphi_l)\Big]\\
&= \sum_{k,l=1}^n ([-\operatorname{Re}\mathbb{A} + z_k + \bar{z}_l]\varphi_k,\varphi_l)\\
&= \sum_{k,l=1}^n \Bigg(\frac{(\operatorname{Re}\mathbb{A} - \bar{z}_l I)^{-1}(\bar{z}_l(\operatorname{Re}\mathbb{A} - \bar{z}_l I) - z_k(\operatorname{Re}\mathbb{A} - z_k I))(\operatorname{Re}\mathbb{A} - z_k I)^{-1}}{z_k\bar{z}_l(z_k - \bar{z}_l)}\\
&\qquad \times Kh_k, Kh_l\Bigg)\\
&= \sum_{k,l=1}^n \left(\frac{\bar{z}_l K^*(\operatorname{Re}\mathbb{A} - z_k I)^{-1}K - z_k K^*(\operatorname{Re}\mathbb{A} - \bar{z}_l I)^{-1}K}{z_k\bar{z}_l(z_k - \bar{z}_l)} h_k, h_l\right)\\
&= \sum_{k,l=1}^n \left(\frac{\bar{z}_l V_\Theta(z_k) - z_k V_\Theta(\bar{z}_l)}{z_k\bar{z}_l(z_k - \bar{z}_l)} h_k, h_l\right).
\end{aligned}$$

The last line can be re-written and hence

$$\sum_{k,\ell=1}^n (\operatorname{Re}\tilde{\mathbb{A}}\,\varphi_k,\varphi_\ell) = \sum_{k,l=1}^n \left(\frac{V_\Theta(z_k)/z_k - V_\Theta(\bar{z}_l)/\bar{z}_l}{z_k - \bar{z}_l} h_k, h_l\right). \tag{17}$$

It is easy too see that $\operatorname{Im}\tilde{\mathbb{A}} = -\operatorname{Im}\mathbb{A} = -KK^*$ and thus (15) yields

$$\begin{aligned}
\sum_{k,\ell=1}^n (\operatorname{Re}\tilde{\mathbb{A}}\,\varphi_k,\varphi_l) &\geq (\cot\alpha)\Big|\sum_{k,\ell=1}^n (\operatorname{Im}\tilde{\mathbb{A}}\,\varphi_k,\varphi_\ell)\Big| = (\cot\alpha)\Big|\sum_{k,\ell=1}^n (KK^*\varphi_k,\varphi_l)\Big|\\
&= (\cot\alpha)\Big|\sum_{k,\ell=1}^n (KK^*\frac{1}{z_k}(\operatorname{Re}\mathbb{A} - z_k I)^{-1}Kh_k, \frac{1}{\bar{z}_l}(\operatorname{Re}\mathbb{A} - \bar{z}_\ell I)^{-1}Kh_l)\Big| \qquad (18)\\
&= (\cot\alpha)\Big|\sum_{k,\ell=1}^n \Big(\frac{V^*(z_l)}{\bar{z}_l}\frac{V(z_k)}{z_k}h_k, h_l\Big)\Big|.
\end{aligned}$$

Combining the above inequality with (17) we obtain (12).

To prove the converse statement we recall that it was shown in [1] that for a minimal system Θ whose symmetric operator $\dot{A}$ is prime, we have

$$\underset{z\neq\bar{z}}{c.l.s.}\,\mathfrak{N}_z = \mathcal{H},$$

and, consequently, $\underset{z\neq\bar{z}}{c.l.s.}\,\mathfrak{N}_z = \mathcal{H}_+$. Thus, (17) and (18) will imply (15). The rest of the converse statement is proved by reversing the argument. Thus we have shown that (12) is equivalent to (13) and this completes the proof. □

Another class that we would like to introduce at this point is a special subclass of scalar realizable inverse Stieltjes functions. Let

$$0 \leq \alpha_1 < \alpha_2 \leq \frac{\pi}{2}.$$

We say that a scalar inverse Stieltjes function $V(z)$ of the class $S_0^{-1}(R)$ belongs to the **class** S^{-1,α_1,α_2} if

$$\tan(\pi - \alpha_1) = \lim_{x\to 0} V(x), \quad \tan(\pi - \alpha_2) = \lim_{x\to-\infty} V(x). \tag{19}$$

The following theorem provides a connection between the classes $S^{-1,\alpha}$ and S^{-1,α_1,α_2}.

Theorem 9. *Let Θ be a scattering accumulative L-system of the form*

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix} \tag{20}$$

with a densely defined non-negative symmetric operator $\dot{A}$. Let also $\tilde{\mathbb{A}}$ of the form (11) *be α-sectorial. Then*

(1) *the impedance function $V_\Theta(z)$ defined by* (7) *belongs to the class S^{-1,α_1,α_2},*
(2) *the operator T of Θ is $(\alpha_2-\alpha_1)$-sectorial with the exact angle of sectoriality $(\alpha_2-\alpha_1)$,*
(3) $\tan\alpha_2 \le \tan\alpha$.

Proof. Since $\tilde{\mathbb{A}}$ is α-sectorial operator, then (13) holds and we can apply Theorem 8 to get

$$\frac{\operatorname{Im}(V_\Theta(z)/z)}{\operatorname{Im} z} \ge (\cot\alpha)\frac{V_\Theta^*(z)V_\Theta(z)}{|z|^2}. \tag{21}$$

Consider the following steps:

$$\begin{aligned}
\frac{\operatorname{Im}(V_\Theta(z)/z)}{\operatorname{Im} z} &= K^*\,\frac{\bar z(\operatorname{Re}\mathbb{A}-\bar zI)^{-1} - z(\operatorname{Re}\mathbb{A}-zI)^{-1}}{z\bar z(\bar z - z)}K \\
&= K^*\,\frac{(\operatorname{Re}\mathbb{A}-\bar zI)^{-1}[\bar z(\operatorname{Re}\mathbb{A}-\bar zI) - z(\operatorname{Re}\mathbb{A}-zI)](\operatorname{Re}\mathbb{A}-zI)^{-1}}{|z|^2(\bar z - z)}K \\
&= K^*\,\frac{(\operatorname{Re}\mathbb{A}-\bar zI)^{-1}[(\bar z - z)\operatorname{Re}\mathbb{A} + (z^2-\bar z^2)I](\operatorname{Re}\mathbb{A}-zI)^{-1}}{|z|^2(\bar z - z)}K \\
&= K^*\,\frac{(\operatorname{Re}\mathbb{A}-\bar zI)^{-1}[-\operatorname{Re}\mathbb{A} + zI + \bar zI](\operatorname{Re}\mathbb{A}-zI)^{-1}}{|z|^2}K,
\end{aligned} \tag{22}$$

that can be checked directly. Using (22) we obtain

$$\lim_{z\to x}\frac{\operatorname{Im}(V_\Theta(z)/z)}{\operatorname{Im} z} = \frac{1}{x^2}K^*(\operatorname{Re}\mathbb{A}-\bar zI)^{-1}[-\operatorname{Re}\mathbb{A}+2xI](\operatorname{Re}\mathbb{A}-zI)^{-1}K. \tag{23}$$

Here we used the fact that $\mathbb{A}$ is accumulative and $x<0$ is a regular point for the quasi-kernel of $\operatorname{Re}\mathbb{A}$. Consequently, (23) yields for $x<0$

$$\begin{aligned}
&-V_\Theta(x) - x^2\lim_{z\to x}\frac{\operatorname{Im}(V_\Theta(z)/z)}{\operatorname{Im} z} = -K^*(\operatorname{Re}\mathbb{A}-xI)^{-1}K \\
&- K^*(\operatorname{Re}\mathbb{A}-xI)^{-1}[-\operatorname{Re}\mathbb{A}+2xI](\operatorname{Re}\mathbb{A}-xI)^{-1}K \\
&= -K^*(\operatorname{Re}\mathbb{A}-xI)^{-1}[I + (-\operatorname{Re}\mathbb{A}+2xI)(\operatorname{Re}\mathbb{A}-xI)^{-1}]K \\
&= -K^*(\operatorname{Re}\mathbb{A}-xI)^{-1}[(\operatorname{Re}\mathbb{A}-xI)(\operatorname{Re}\mathbb{A}-xI)^{-1} + (-\operatorname{Re}\mathbb{A}+2xI)(\operatorname{Re}\mathbb{A}-xI)^{-1}]K \\
&= -K^*(\operatorname{Re}\mathbb{A}-xI)^{-1}[\operatorname{Re}\mathbb{A}-xI-\operatorname{Re}\mathbb{A}+2xI](\operatorname{Re}\mathbb{A}-xI)^{-1}K \\
&= -K^*(\operatorname{Re}\mathbb{A}-xI)^{-1}[xI](\operatorname{Re}\mathbb{A}-xI)^{-1}K \\
&= -xK^*(\operatorname{Re}\mathbb{A}-xI)^{-1}(\operatorname{Re}\mathbb{A}-xI)^{-1}K \\
&= -xK^*(\operatorname{Re}\mathbb{A}-xI)^{-1}KK^*(\operatorname{Re}\mathbb{A}-xI)^{-1}K = -x(V_\Theta(x))^2 \ge 0.
\end{aligned}$$

Therefore

$$-V_\Theta(x) \ge x^2\lim_{z\to x}\frac{\operatorname{Im}(V_\Theta(z)/z)}{\operatorname{Im} z} \quad (x<0) \tag{24}$$

or

$$-\frac{V_\Theta(x)}{x^2} \ge \lim_{z\to x}\frac{\operatorname{Im}(V_\Theta(z)/z)}{\operatorname{Im} z}. \tag{25}$$

Thus, using (21) and (25) we get

$$-V_\Theta(x) \geq (\cot\alpha) V_\Theta^2(x) \quad (x<0), \tag{26}$$

and therefore

$$-V_\Theta(x) \leq \tan\alpha \quad (x<0). \tag{27}$$

It follows from Theorem 6 that the impedance function $V_\Theta(z)$ of an L-system with an accumulative operator $\mathbb{A}$ has the integral representation (9) with $\beta = 0$, i.e.,

$$V_\Theta(z) = \gamma + \int_0^\infty \left(\frac{1}{t-z} - \frac{1}{t}\right) dG(t). \tag{28}$$

Then (27) and (28) yield

$$-V_\Theta(x) = -\gamma - \int_0^\infty \left(\frac{1}{t-x} - \frac{1}{t}\right) dG(t) \leq \tan\alpha \quad (x<0,\ |x|>1)$$

and thus

$$\int_0^\infty \frac{dG(t)}{t} < \infty \quad \text{and} \quad -\gamma - \int_0^\infty \frac{dG(t)}{t} \leq \tan\alpha.$$

Let us denote

$$\tan(\pi-\alpha_1) = \gamma, \quad \tan(\pi-\alpha_2) = \gamma - \int_0^\infty \frac{dG(t)}{t}. \tag{29}$$

Using (29) we obtain that $V_\Theta(z) \in S^{-1,\alpha_1,\alpha_2}$ and $\tan\alpha_2 \leq \tan\alpha$.

It was shown in [1] (see Theorem 8.2.4) for the system Θ of the form (20) there is a system

$$\Theta' = \begin{pmatrix} S & \sqrt{2}(I+\mathbb{A})^{-1}K & -1 \\ \mathcal{H} & & \mathbb{C} \end{pmatrix} \tag{30}$$

with the main operator $S = (I-T)(I+T)^{-1}$ and such that

$$W_\Theta(z) = W_\Theta(-1) W_{\Theta'}\left(\frac{1-z}{1+z}\right), \quad z\in\rho(T), \quad z\neq -1. \tag{31}$$

We know from [1] (see Lemma 9.5.12) that T is α-sectorial if and only if S is α-co-sectorial contraction, i.e.,

$$\|S\sin\alpha \pm iI\cos\alpha\| \leq 1.$$

It is also shown in [1] that the exact angle of co-sectoriality of S can be calculated via

$$\cot\beta = \frac{1 + V_{\Theta'}(1)V_{\Theta'}(-1)}{|V_{\Theta'}(-1) - V_{\Theta'}(1)|}. \tag{32}$$

Let us compute $V_{\Theta'}(1)$ and $V_{\Theta'}(-1)$ using (31) and (2). We get

$$V_{\Theta'}(1) = -i(I + W_\Theta^{-1}(-1)W_\Theta(0))^{-1}(W_\Theta^{-1}(-1)W_\Theta(0) - I),$$
$$V_{\Theta'}(-1) = -i(I + W_\Theta^{-1}(-1)W_\Theta(-\infty))^{-1}(W_\Theta^{-1}(-1)W_\Theta(-\infty) - I)$$

and

$$W_\Theta(0) = \frac{1-iV_\Theta(0)}{1+iV_\Theta(0)}, W_\Theta(-\infty) = \frac{1-iV_\Theta(-\infty)}{1+iV_\Theta(-\infty)}, W_\Theta^{-1}(-1) = \frac{1+iV_\Theta(-1)}{1-iV_\Theta(-1)}.$$

This yields

$$V_{\Theta'}(1) = \frac{V_\Theta(-1) - V_\Theta(-\infty)}{1 + V_\Theta(-1)V_\Theta(-\infty)}, \quad V_{\Theta'}(-1) = \frac{V_\Theta(-1) - V_\Theta(0)}{1 + V_\Theta(-1)V_\Theta(0)}.$$

Taking into account (32) we get

$$\cot\beta = \frac{1+V_\Theta(0)V_\Theta(-\infty)}{V_\Theta(0)-V_\Theta(-\infty)} = \frac{1+\tan(\pi-\alpha_1)\cdot\tan(\pi-\alpha_2)}{\tan(\pi-\alpha_1)-\tan(\pi-\alpha_2)} = \cot(\alpha_2-\alpha_1).$$

□

Note that Theorem 9 also remains valid for the case when the operator $\tilde{\mathbb{A}}$ is accretive but not α-sectorial for any $\alpha\in(0,\pi/2)$.

The corollary below treats the case when α in Theorem 9 is the exact angle of sectoriality of the operator T. Thus both operators T and $\mathbb{A}$ maintain the same exact angle.

Corollary 10. *Let Θ of the form* (20) *be an L-system as in the statement of Theorem* 9 *and let α be the exact angle of sectoriality of the operator T of Θ. Then $V_\Theta(z)\in S^{-1,0,\alpha}$.*

Proof. According to Theorem 9 the exact angle of sectoriality is given by $\alpha_2-\alpha_1$, where $\tan\alpha_1$ and $\tan\alpha_2 = \lim_{x\to-0}V_\Theta(x)$ are derived from (19). It was also shown that $\tan\alpha\ge\tan\alpha_2$. On the other hand, since in the statement of the current corollary α is the exact angle of sectoriality of T, then $\alpha=\alpha_2-\alpha_1$ and hence $\tan(\alpha_2-\alpha_1)\ge\tan\alpha_2$. Therefore, $\alpha_1=0$. □

Remark 11. *It follows that under assumptions of Corollary* 10*, the impedance function $V_\Theta(z)$ has the form*

$$V_\Theta(z) = \int_0^\infty\left(\frac{1}{t-z}-\frac{1}{t}\right)dG(t).$$

Theorem 12. *Let Θ be a minimal accumulative L-system of the form* (20)*, where $\dot{A}$ is a closed densely defined non-negative symmetric operator with deficiency numbers* $(1,1)$*. Let also $\tilde{\mathbb{A}}$ be defined via* (11)*. If the impedance function $V_\Theta(z)$ belongs to the class S^{-1,α_1,α_2}, then $\tilde{\mathbb{A}}$ is α-sectorial, where*

$$\tan\alpha = \tan\alpha_2 + 2\sqrt{\tan\alpha_1(\tan\alpha_2-\tan\alpha_1)}. \tag{33}$$

Proof. Since $V_\Theta(z)\in S^{-1,\alpha_1,\alpha_2}$, we use (28) and (29) to get

$$V_\Theta(z) = -\tan\alpha_1+\int_0^\infty\left(\frac{1}{t-z}-\frac{1}{t}\right)dG(t)\quad\text{and}\quad\tan\alpha_2=\tan\alpha_1+\int_0^\infty\frac{dG(t)}{t}. \tag{34}$$

It is easily seen from (34) that

$$V_\Theta(z)/z = -\frac{1}{z}\tan\alpha_1+\int_0^\infty\frac{dG(t)}{t(t-z)}. \tag{35}$$

Let $\{z_k\}$, $k=1,\dots,n$ be an arbitrary numbers in $\mathbb{C}_+$ and $\{\xi_k\}$, $k=1,\dots,n$ be arbitrary complex numbers. By direct substitution and using (34)–(35) one gets

$$\sum_{k,l=1}^n\frac{V_\Theta(z_k)/z_k-V_\Theta(\bar z_l)/\bar z_l}{z_k-\bar z_l}\bar\xi_k\xi_l = (\tan\alpha_1)\left|\sum_{l=1}^n\frac{\xi_l}{z_l}\right|^2+\int\limits_0^\infty\left|\sum_{l=1}^n\frac{\xi_l}{\sqrt{t}(t-z_l)}\right|^2dG(t). \tag{36}$$

Furthermore, it follows from (34) and (35) that

$$\begin{aligned}\sum_{k,l=1}^n\frac{V_\Theta(z_k)}{z_k}\frac{V_\Theta(\bar z_l)}{\bar z_l}\xi_l\bar\xi_k &= \sum_{k,l=1}^n\left(-\frac{1}{z_k}\tan\alpha_1+\int_0^\infty\left(\frac{1}{t-z_k}-\frac{1}{t}\right)dG(t)\right)\\ &\times\left(-\frac{1}{\bar z_l}\tan\alpha_1+\int_0^\infty\left(\frac{1}{t-\bar z_l}-\frac{1}{t}\right)dG(t)\right)\xi_l\bar\xi_k\\ &=\left|\sum_{l=1}^n(-\tan\alpha_1)\frac{\xi_l}{z_l}+\int_0^\infty\sum_{l=1}^n\left(\frac{\xi_l}{\sqrt{t}(t-z_l)}\right)dG(t)\right|^2.\end{aligned} \tag{37}$$

Applying (37) we have that

$$\begin{aligned}
&\left|\sum_{l=1}^{n}(-\tan\alpha_1)\frac{\xi_l}{z_l}+\int_0^\infty\sum_{l=1}^{n}\left(\frac{\xi_l}{\sqrt{t}(t-z_l)}\right)dG(t)\right|\le\tan\alpha_1\left|\sum_{l=1}^{n}\frac{\xi_l}{z_l}\right|+\left|\int_0^\infty\sum_{l=1}^{n}\frac{\xi_l dG(t)}{\sqrt{t}(t-z_l)}\right|\\
&\quad=\tan\alpha_1\left|\sum_{l=1}^{n}\frac{\xi_l}{z_l}\right|+\left|\int_0^\infty\frac{1}{\sqrt{t}}\sum_{l=1}^{n}\frac{\xi_l dG(t)}{t-z_l}\right|\\
&\quad\le\tan\alpha_1\left|\sum_{l=1}^{n}\frac{\xi_l}{z_l}\right|+\left(\int_0^\infty\frac{dG(t)}{t}\right)^{1/2}\left(\int_0^\infty\left|\sum_{l=1}^{n}\frac{\xi_l}{t-z_l}\right|^2dG(t)\right)^{1/2}\\
&\quad\le(\tan\alpha_1)^{1/2}\left[\tan\alpha_1\left|\sum_{l=1}^{n}\frac{\xi_l}{z_l}\right|^2+\int_0^\infty\left|\sum_{l=1}^{n}\frac{\xi_l}{t-z_l}\right|^2dG(t)\right]^{1/2}\\
&\quad+\left(\int_0^\infty\frac{dG(t)}{t}\right)^{1/2}\left[\tan\alpha_1\left|\sum_{l=1}^{n}\frac{\xi_l}{z_l}\right|^2+\int_0^\infty\left|\sum_{l=1}^{n}\frac{\xi_l}{t-z_l}\right|^2dG(t)\right]^{1/2}\\
&\quad=\left[\tan^{1/2}\alpha_1+\left(\int_0^\infty\frac{dG(t)}{t}\right)^{1/2}\right]\left(\sum_{k,l=1}^{n}\frac{V_\Theta(z_k)/z_k-V_\Theta(\bar z_l)/\bar z_l}{z_k-\bar z_l}\bar\xi_k\xi_l\right)^{1/2}.
\end{aligned}$$

Using (36), (37) we obtain

$$\begin{aligned}
\sum_{k,l=1}^{n}\frac{V_\Theta(z_k)}{z_k}\frac{V_\Theta(\bar z_l)}{\bar z_l}\xi_l\bar\xi_k&\le\left[\tan^{1/2}\alpha_1+\left(\int_0^\infty\frac{dG(t)}{t}\right)^{1/2}\right]^2\\
&\quad\times\sum_{k,l=1}^{n}\frac{V_\Theta(z_k)/z_k-V_\Theta(\bar z_l)/\bar z_l}{z_k-\bar z_l}\bar\xi_k\xi_l.
\end{aligned}\tag{38}$$

This implies that $V_\Theta(z)$ belongs to the class $S^{-1,\alpha}$ with

$$\begin{aligned}
\tan\alpha&=\left[\tan^{1/2}\alpha_1+\left(\int_0^\infty\frac{dG(t)}{t}\right)^{1/2}\right]^2\\
&=\tan\alpha_1+(\tan\alpha_2-\tan\alpha_1)+2\sqrt{\tan\alpha_1}\cdot\sqrt{\tan\alpha_2-\tan\alpha_1}\\
&=\tan\alpha_2+2\sqrt{\tan\alpha_1}\cdot\sqrt{\tan\alpha_2-\tan\alpha_1}.
\end{aligned}$$

Applying Theorem 8 we get that $\tilde{\mathbb{A}}$ is α-sectorial with the angle α described by formula (33). □

The next statement gives an explicit description of all the functions from the class S^{-1,α_1,α_2} that are realizable as impedance functions of such L-systems that the exact angles of sectoriality of T and $\tilde{\mathbb{A}}$ coincide. Its proof immediately follows from Theorems 9 and 12.

Theorem 13. *Let Θ be an L-system of the form* (20) *from the statement of Theorem* 12. *Then both $\tilde{\mathbb{A}}$ and $T\in\Lambda(\dot A)$ are α-sectorial operators with the exact angle $\alpha\in(0,\pi/2)$ if and only if*

$$V_\Theta(z)=\int_0^\infty\left(\frac{1}{t-z}-\frac{1}{t}\right)\in S^{-1,0,\alpha}.$$

Moreover, the angle α can be found via the formula

$$\tan\alpha=\int_0^\infty\frac{dG(t)}{t}.\tag{39}$$

Therefore, we have shown that within the conditions of Theorem 13 the α-sectorial sesquilinear form (f, Tf) defined on a subspace $\mathrm{Dom}(T)$ of $\mathcal{H}_+$ can be extended to the α-sectorial form $(\tilde{\mathbb{A}}f, f)$ defined on $\mathcal{H}_+$ preserving the exact (for both forms) angle of sectoriality α. A general problem of extending sectorial sesquilinear forms was mentioned by T. Kato in [6].

5. Examples

Example 1. Consider a function

$$V(z) = i\sqrt{z}.$$

A direct check (see also [3] and [7]) confirms that $V(z)$ is an inverse Stieltjes function. Clearly, $V(z)$ belongs to the class $S^{-1,0,\pi/2}$.

Let us consider a symmetric operator given by

$$\begin{cases} \dot{A}y = -y'', \\ y(0) = y'(0) = 0. \end{cases} \tag{40}$$

Then its adjoint operator $\dot{A}^*$ is defined in $L_2[0,+\infty)$ by $\dot{A}^*y = -y''$ without any boundary conditions. It was shown in [1] that we can construct an L-system Θ with Schrödinger operator based on $\dot{A}$ of the form (40) that realizes $V(z)$

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset L_2[0,+\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}, \tag{41}$$

where

$$\mathbb{A}\, y = -y'' - [iy'(0) + y(0)]\delta'(x) \tag{42}$$

and operator T

$$\begin{cases} Ty = -y'', \\ y'(0) = iy(0). \end{cases} \tag{43}$$

The space $\mathcal{H}_+$ in the above system was constructed with operator $\dot{A}$ of the form (40) using (3). Also, $\delta(x) \in \mathcal{H}_-$ and $\delta'(x) \in \mathcal{H}_-$ are delta function and its derivative such that $(y, \delta) = y(0)$ and $(y, \delta') = -y'(0)$.

The channel operator is given by $Kc = cg$, $g = \delta'(x)$, $(c \in \mathbb{C})$ (see [1]) with

$$K^*y = (y, g) = -y'(0). \tag{44}$$

Consider also operator $\tilde{\mathbb{A}}$ defined by (11). We have

$$\tilde{\mathbb{A}}y = 2\,\mathrm{Re}\,\dot{A}^*y - \mathbb{A}y = -y'' - y'(0)\delta(x) - y(0)\delta'(x) + [iy'(0) + y(0)]\delta'(x).$$

One can see that operator T of the from (43) is accretive but not α-sectorial for any α and so is operator $\tilde{\mathbb{A}}$ above.

All the derivations above can be repeated for an inverse Stieltjes function of the form

$$V(z) = \gamma + i\sqrt{z}, \quad -\infty < \gamma \le 0,$$

with very minor changes. Clearly, $V(z) \in S^{-1,-\gamma,\pi/2}$. In this case (see [1]) the operator T has a form

$$\begin{cases} Ty = -y'', \\ y'(0) = (-\gamma + i)y(0). \end{cases} \tag{45}$$

This operator is α-sectorial with

$$\tan\alpha = -\frac{1}{\gamma}. \tag{46}$$

The state-space operator $\mathbb{A}$ of the realizing system in this case is

$$\mathbb{A}\, y = -y'' - [iy'(0) + (1 + i\gamma)y(0)][-\gamma\delta(x) + \delta'(x)],$$

and operator K is again of the form $Kc = cg$, $(c \in \mathbb{C})$, where $g = [-\gamma\delta(x) + \delta'(x)]$. The realizing system Θ has the form (41) with these operators. The operator $\tilde{\mathbb{A}}$ defined by the form

$$\tilde{\mathbb{A}}y = -y'' - y'(0)\delta(x) - y(0)\delta'(x) + [iy'(0) + (1+i\gamma)y(0)][-\gamma\delta(x) + \delta'(x)]$$

is accretive but not α-sectorial for any α.

Example 2. Consider a function

$$V(z) = -\frac{\sqrt{z}}{\sqrt{z} + 2i}. \tag{47}$$

Running a direct check similar to Example 1 confirms that $V(z)$ is an inverse Stieltjes function. It is also easy to see that $V(z)$ belongs to the class $S^{-1,0,\pi/4}$. Now we will assemble an L-system Θ of the form

$$\Theta = \begin{pmatrix} \mathbb{A} & K & 1 \\ \mathcal{H}_+ \subset L_2[0,+\infty) \subset \mathcal{H}_- & & \mathbb{C} \end{pmatrix}$$

based on a Schrödinger operator such that its impedance function $V_\Theta(z)$ coincides with $V(z)$. Let $\dot{A}$ be defined by (40) and

$$\begin{cases} Ty = -y'', \\ y'(0) = (1+i)y(0). \end{cases} \tag{48}$$

It follows from [1] that T of the form (48) is α-sectorial with the exact angle $\alpha = \pi/4$. Using the results from [1] we set

$$\mathbb{A}\, y = -y'' - \frac{1}{1+i}[y'(0) - (1+i)y(0)]\delta'(x)$$

and use the operator K given by $Kc = cg$, $(c \in \mathbb{C})$, where $g = \frac{1}{\sqrt{2}}\delta'(x)$, to obtain the system Θ of the form (41). The associated operator $\tilde{\mathbb{A}}$ then is

$$\tilde{\mathbb{A}}y = -y'' - y'(0)\delta(x) - y(0)\delta'(x) + \frac{1}{1+i}[y'(0) - (1+i)y(0)]\delta'(x).$$

By direct calculations we obtain that

$$(\operatorname{Re}\tilde{\mathbb{A}}y, y) = \|y'(x)\|^2_{L^2} + \frac{1}{2}|y'(0)|^2, \quad (\operatorname{Im}\tilde{\mathbb{A}}y, y) = -\frac{1}{2}|y'(0)|^2,$$

and hence

$$(\operatorname{Re}\tilde{\mathbb{A}}y, y) \geq |(\operatorname{Im}\tilde{\mathbb{A}}y, y)|.$$

Thus, $\tilde{\mathbb{A}}$ is α-sectorial with $\alpha = \pi/4$. According to Theorem 13 this angle of sectoriality is exact.

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